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# SLOW DYNAMICS AT QUANTUM PHASE TRANSITIONS

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This paper discusses how slow dynamics can occur in the vicinity of quantum phase transitions. In addition to critical slowing down, there are two additional sources of slow dynamics. Firstly, even for a pure system, there is a phase coherence time which diverges as the temperature tends to zero. This is hard to see from numerical studies in imaginary time and real time dynamics are needed. Secondly, in random systems, Griffiths-McCoy singularities occur at low temperature because of rare regions which are "locally in the wrong phase" due to of statistical fluctuations in the random interactions. These are particularly strong for systems with discrete, e.g. Ising, symmetry. The form of Griffiths-McCoy singularities in real time is not understood.

## 1. What is a Quantum Phase Transition?

In books on critical phenomena one rarely sees any mention of Planck's constant. This is because quantum mechanics is irrelevant close to the critical point if the transition temperature, $T_c$ is finite. To see this, compare the thermal energy per degree of freedom, $k_B T$, with the zero point quantum energy, $\hbar\omega$, where $\omega$ is the frequency of the fluctuation. We are interested in $T$ close to $T_c$ and $\omega$ equal to the characteristic frequency of the critical fluctuations, $\omega_c$. The latter vanishes as the critical point is approached because of "critical slowing down", and so, near the critical point we must have

$$\hbar\omega_c \ll k_B T_c, \tag{1}$$

which implies that quantum effects are unimportant. However, if we have some parameter in the system, which we will generically call $\Delta$, which decreases $T_c$, then this condition is harder and harder to satisfy as $T_c$ gets smaller, i.e. the size of this "classical" critical region gets smaller. If we were to cross the transition at zero temperature, by changing $\Delta$, then there are no thermal fluctuations at all, quantum mechanics *is* relevant and we have what is called a quantum critical point[1].

Interest in quantum critical points arises in part because they are in different universality classes from the classical, thermally driven transitions which occur at finite-$T$. In addition, in the presence of disorder, they remain poorly understood despite a lot of effort both experimentally and theoretically. Examples of quantum transitions with disorder are (i) the Anderson localization problem[2], and (ii) transitions between quantum Hall states in two dimensional electron systems in a magnetic field[1] For Anderson localization, one adds defects to a metal until, at some concentration, it becomes an insulator at zero temperature because the electron states are localized. One can measure how the conductivity, extrapolated to zero temperature, varies with concentration. For the quantum Hall case, one can change either the magnetic field or the carrier concentration to go from a state with one value of the quantized Hall conductance to a state with a different value. In both cases, many experiments have found that the expected scaling behavior does not seem to work[3,4], perhaps because the critical region only occurs at unexpectedly small temperatures. Theoretically, our understanding, in particular our ability to determine the universality classes, is hampered by the failure here of the perturbative renormalization group approach. One finds that the effective value of the disorder increases at large scales and the parameters "flow" outside the region where perturbation theory is valid. This may be related to non-perturbative effects called Griffiths-McCoy singularities[5,6], which we shall discuss later, which occur even away from the critical point.

Given this poor understanding it is useful to study the simplest appropriate model, which we define and discuss in Sec. 2.

Since this conference is on conquering long time scales I will focus the lecture on how such long time scales arise near quantum phase transitions and on problems involved in computing properties at these long times. It is well known that timescales become very long close to a critical point because of critical slowing down. This is characterized by a dynamical exponent, $z$, relating the characteristic time scale $\tau$ to the correlation length $\xi$ by $\tau \sim \xi^z$. There are, however, other sources of slow dynamics near a quantum critical point, and these, rather than critical dynamics, will be the focus of the talk. First of all, the system goes into a unique quantum state at $T = 0$ and as $T \to 0$ there is a phase coherence time, $\tau_\phi$, which diverges. We shall see that this is essentially impossible to extract from numerical simulations in imaginary time, and a means of doing real time dynamics of an interacting quantum system at finite temperature is required. Unfortunately, there are not systematic numerical methods for doing this, and we shall focus on the special case of one-dimension, where some tricks can be used. This is discussed in Sec. 3.

A diverging $\tau_\phi$ occurs even for a pure system and is most conveniently studied for that case. However, the other source of long time scales that we discuss, Griffiths-McCoy[5,6] singularities, only arises in the presence of disorder. This is discussed in Sec. 4. Finally we give some brief concluding remarks in Sec. 5.

## 2. The Model

As discussed in the previous section, quantum phase transitions with disorder are poorly understood. Furthermore, even for pure systems, real time dynamics at long times are also not well understood. It is therefore sensible to study the simplest model with a quantum phase transition to try to understand it in detail. Just as the simplest model with a classical transition is the Ising model, we argue that an Ising model made quantum by adding a transverse field is the simples model with a quantum phase transition. The Hamiltonian is

$$\mathcal{H} = \sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z - \sum_i h_i \sigma_i^x \ , \qquad (2)$$

where the $\{\sigma_i^\alpha\}$ are Pauli spin matrices, the $J_{ij}\,(>0)$ are nearest neighbor interactions and the $h_i$ are transverse fields. For some of this paper we will consider a pure model where all the $h_i$ are equal to $h$, and the $J_{ij}$ are also equal. In other sections we will take both the $J_{ij}$ and the $h_i$ to be independent random variables.

If the $h_i$ are all zero we have the standard classical Ising model with all spins parallel in the ground state. Furthermore, for dimension $d$ greater than one, this long range ferromagnetic order persists to finite temperature. Turning on the transverse fields, the strength of the order in the ground state is diminished by quantum fluctuations until it eventually disappears at the quantum critical point at $h = h_c, T = 0$. In $d > 1$ the value of $T_c$ decreases to zero at the same point.

Frequently, this model is studied by quantum Monte Carlo fluctuations in which case it is transformed into an effective classical action in $d+1$ dimensions, where the extra dimension is imaginary time, lying in the interval from 0 to $\beta \equiv 1/k_B T$. In principle real time dynamics can be obtained by analytic continuation from imaginary time correlations, but this is, in general, quite difficult, and essentially impossible from numerical data at long times where $\hbar/t \ll k_B T$.

## 3. Phase Coherence Time

This section is based on work with Subir Sachdev[7], referred to as SY. Consider the pure transverse Ising model. Away from the critical point it has a gap in the excitation spectrum. There is a *finite* time scale equal to the inverse of this gap. However, in addition, there is a time scale which diverges in the zero temperature limit, the phase coherence time.

To investigate this in detail, SY considered the $d = 1$ case where there is only long range order at zero temperature and for transverse field strength less than $h_c$. We shall concentrate on the so-called "renormalized classical" (RC) region for $T \to 0$, $h < h_c$, where ferromagetic order arises at zero temperature.

At zero temperature there are well defined excitations above the ground state with energy $\epsilon_p$, where $p$ is the momentum. At higher energy there are continua of multi-particle states. In the RC region, these excitations are domain walls separating a region in which spins are up from a region in which they are down. At finite temperature there is a finite population of these excitations. The big simplification is that these excitations can be treated *classically*, since their average separation, which is the correlation length, diverges *exponentially* as $T \to 0$, while their thermal de Broglie wavelength varies only as $T^{-1/2}$ and so is much smaller. As a result, SY find that in the low-$T$ RC regime, and also in the corresponding regime for $h > h_c$, the spin correlation function can be expressed as a product of two factors. One, arising from quantum effects, gives the $T = 0$ value of the correlation function, and the other, which comes from a *classical* theory, describes the effects of temperature. In the RC region, SY find that

$$C(x,t) \equiv \langle \sigma_x^z(t)\sigma_0^z(0)\rangle = N_0^2 R(x,t) \qquad \text{(RC region)}, \tag{3}$$

where $N_0$ is the magnetization at zero temperature, reduced from unity by quantum fluctuations, and

$$R(x,t) = \exp\left(-\int \frac{dp}{\pi} e^{-\epsilon_p/T} \left|x - v_p t\right|\right). \tag{4}$$

The expression for $R(x,t)$ comes from an argument involving classical statistics.

For $t = 0$ we have

$$R(x,0) = e^{-|x|/\xi} \tag{5}$$

where $\xi = (\pi/2mT)^{1/2} e^{mc^2/T}$ diverges exponentially as $T \to 0$, as expected in one-dimension. Similarly for $x = 0$,

$$R(0,t) = e^{-|t|/\tau_\phi}, \tag{6}$$

where $\tau_\phi = (\pi/2T) e^{mc^2/T}$, the phase coherence time, also diverges exponentially. However, for arbitrary $x$ and $t$ the behavior is more complicated than a single exponential.

SY verified this expression numerically by a mapping of the nearest neighbor transverse Ising chain to free fermions[8,9], generalized to calculations of time-dependent[7,10,11] properties. They find that in the RC regime the imaginary part of $C(x,t)$ is much smaller than its real part, which suggests that the dynamics is indeed classical. Fig. 1 shows data for the real part for $g(= h/h_c) = 0.6, T = 0.3$ and $x = 20$ for a lattice size $L = 256$. For comparison we show the theoretical prediction from Eqs. (3) and (4).The agreement is remarkably good. It is interesting to note that because there is a maximum velocity, $v_{max}$, Eq. (4) predicts that $C(x,t)$ should be independent of $t$ for $|t| < |x|/v_{max}$, and the numerical results show this very clearly. The inset, with a much increased vertical scale, gives an idea of how small are the deviations between the theory and numerics.

SY also find in the region $h > h_c, T \to 0$ (often called "quantum diordered"), that $C(x,t)$ again factorizes into the product of the zero temperature value, determined by quantum fluctuations, and a finite temperature factor determined by a classical calculation. The latter turns out to be the same function, $R(x,t)$, found above in the RC region. However, a discussion of the quantum disordered region is rather complicated so we omit it here.

It would be interesting to generalize these calculations to higher dimensions. However, the analytical calculations depended crucially on the dimension being one, and the numerical work did too since the mapping to *free* fermions only works in that case. In general, analytical and numerical methods rely on the imaginary time formalism followed by analytic continuation to real time. However, as emphasized by Sachdev[12] this fails at long times and finite temperatures, more precisely for

$$\frac{\hbar}{t} \ll k_B T \tag{7}$$

because the largest value of imaginary time is $\hbar/k_B T$. There is a need for general numerical methods to investigate real time properties of *interacting* quantum

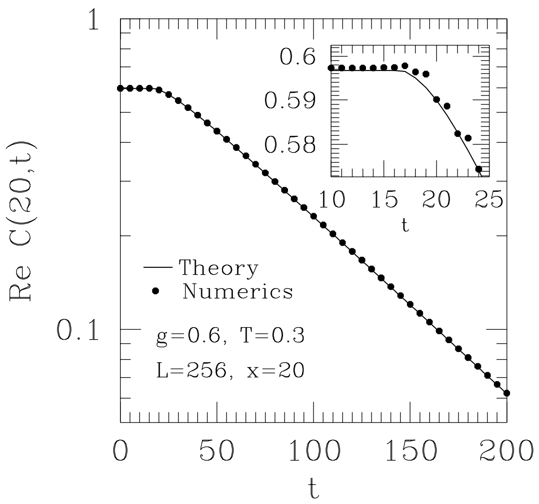


Figure 1: The points show numerical data for the nearest neighbor model, in the RC region, obtained for a lattice size $L = 256$ with free boundary conditions. This is compared with the theoretical prediction in Eqs. (3) and (4).The two sites were chosen to be as close as possible to the center of the lattice.

systems at finite temperature. At present, however, they do not exist. We were able to perform the calculations in one-dimension with nearest neighbor interactions because in that special case, and only in that case, the problem reduces to *non-interacting* particles.

## 4. Griffith-McCoy Singularities

The random transverse field Ising model also has striking singularities in the paramagnetic phase. These are of the type first discussed by Griffiths[5] for a random classical system, but are much more pronounced in the quantum case. They come from local regions in which the interactions happen to be stronger than average and/or the transverse fields are weaker than average, so that these regions can be viewed as "locally in the ferromagnetic phase", even though the whole sample is paramagnetic. Spins in such regions effectively act as one large spin, as a result of which their dynamics is very slow. A simple phenomenological picture[13], leads to the prediction that, at $T = 0$, the average, on-site, time-dependent correlation function, varies as a power of (imaginary) time, with an exponent, $z'(\delta)$, varying continuously with $\delta$, the deviation from the critical point. Similarly, the susceptibility and specific heat are predicted to vary with a power of $T$, which depends on $z'(\delta)$, as $T \to 0$.

This general picture has been confirmed by numerics[9,11,14] both in one-dimension and higher dimensions. If $z'(\delta)$ is sufficiently large (in practice greater than the space dimension, $d$) the average local susceptibility will actually diverge as $T \to 0$. A surprising feature of one-dimension[15,9] is that $z'(\delta) \to \infty$ as $\delta \to 0$, so there will *always* be a region of the paramagnetic phase where the susceptibility diverges. This is quite remarkable; normally the susceptibility only diverges *at* the transition whereas here it diverges for a *range* of $\delta$ around the transition. This comes entirely from the long range (power-law) correlations in time. The correlations in space decay exponentially except precisely at criticality.

Recently it has been argued[14] on the basis of quantum Monte Carlo simulations that a similar divergence of the susceptibility for a range of $\delta$ also holds in two dimensions. Since this shows that $d = 1$ is not special, it is natural to assume that the same will be true in three dimensions, though direct verification by numerics has not yet been done.

We should emphasize that the Griffiths-McCoy singularities are as strong as this because of the discrete (Ising) symmetry. By contrast, for an isotropic quantum rotor model[12] with $n$ components for $n \geq 2$, correlations in imaginary time fall off faster than a power law, probably as a stretched exponential.

We have seen that both phenomenological arguments and numerics show that imaginary time correlation functions decay with a power law in the Griffiths-McCoy phase away from the critical point. An unsolved problem[16] is what the correlation functions look like in real time. As mentioned in the previous section, it is essentially impossible to analytically continue approximate imaginary time functions to real time in the long time limit.

## 5. Conclusions

We have seen that in quantum systems at low temperature, there are long time scales which need to be studied in *real time*. Analytic continuation from *imaginary time* fails at times $t$ such that $\hbar/t \ll k_B T$ (except for an exact solution). Unfortunately, determining real time dynamics of an interacting quantum system at finite temperature remains, in general an *unconquered* problem.

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