Phase Transitions in Random Systems

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Abstract. These lectures discuss the theory of phase transitions and critical phenomena in random systems. First a discussion is given of the basic principles, including averaging, the concept of frustration, and the Harris criterion for deciding whether weak randomness is relevant. This is followed by the study of several examples: weakly random ferromagnets, the percolation problem, the random field problem and spin glasses.

1. Introduction

In these lectures I will discuss the theory of phase transitions [1] in disordered systems. An emphasis will be placed on magnetic systems, but many of the ideas are more general. In the past, randomness was felt to be a *nuisance*, but one that had to be understood since real crystals have defects. Recently, it has been realized that many random systems exhibit properties which are quite different from pure systems, and are *interesting in their own right.* In these lectures on phase transitions in disordered systems I will try to emphasize (i) basic ideas and (ii) unsolved problems. After discussing some of the simple models and the general theory of disordered systems, I next consider phase transitions in random ferromagnets. These are "unfrustrated", *i.e.* there is no competition between different interactions and so the ground state is trivial. Then I will discuss two examples of "frustrated systems", random field systems and spin glasses. Here the nature of the ground state is non-trivial and even the existence of a phase transition in three dimensions has been controversial.

2. Simple Models

The simplest model for phase transitions is the Ising model [1]. If there is no disorder and just nearest neighbor interactions, then the Hamiltonian can be written

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j \ , \tag{1}$$

where $S_i = \pm 1$, the sum is over all nearest neighbor pairs (counted once) and the nearest neighbor interaction strength is J. As the simplest example of a random problem consider *site dilution* where there is a spin on only a fraction p of sites, and the remaining $1 - p$ are empty. The Hamiltonian of the site-dilute problem can be written in a form somewhat similar to that of Eq. (1),

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \epsilon_i \epsilon_j S_i S_j \ , \tag{2}$$

where ϵ_i describes the occupancy of site i and takes the value 1 with probability p and 0 with probability $1-p$. Note the big difference between the variables S_i and ϵ_i. The S_i are *statistical variables* which are *summed over*, while the ϵ_i are *quenched random variables* which are assumed *fixed*. This is because the atoms, whose positions are random, do not move, in practice, during the time of a measurement. The opposite limit, in which the atoms move enough to fully equilibrate during the time of the measurement, is called the annealed problem, and will not be discussed here because it does not usually correspond to the experimental situation.

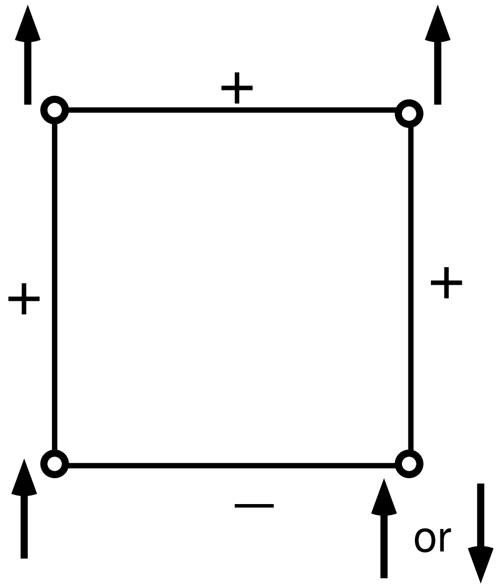

Figure 1. A toy example with four sites which illustrates the concept of frustration. Three of the bonds are ferromagnetic, indicated by a "+" and one is antiferromagnetic, indicated by a "−". No configuration of the Ising spins simultaneously minimizes each term in the Hamiltonian. More generally, the system will be frustrated if there are an odd number of antiferromagnetic interactions around an elementary polygon on the lattice.

In another model one assumes that the disorder is in the *bonds* rather than in the sites. Whereas this is not quite so close to the experimental situation as the site random problem, it is a bit simpler theoretically. For example one could take the nearest neighbor model in Eq. (1) and assume that each interaction is an independent random variable which can take one of two values, *i.e.* the Hamiltonian is

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \ , \tag{3}$$

in which the J_{ij} have the distribution

$$P(J_{ij}) = p\delta(J_{ij} - J_1) + (1-p)\delta(J_{ij} - J_2) \ . \tag{4}$$

We shall call this a *random bond* model. The special case of $J_2 = 0$ will correspond to *bond dilution*. If J_1 and J_2 are both positive the ground state must have all spins parallel (either all up or all down) since this configuration simultaneously minimizes each term in the Hamiltonian. Such a model is said to be *unfrustrated.* However, if J_1 and J_2 have opposite signs then no spin configuration simultaneously minimizes each term in $\mathcal{H}$, as shown for a toy example in Fig. 1. Such a system is said to be

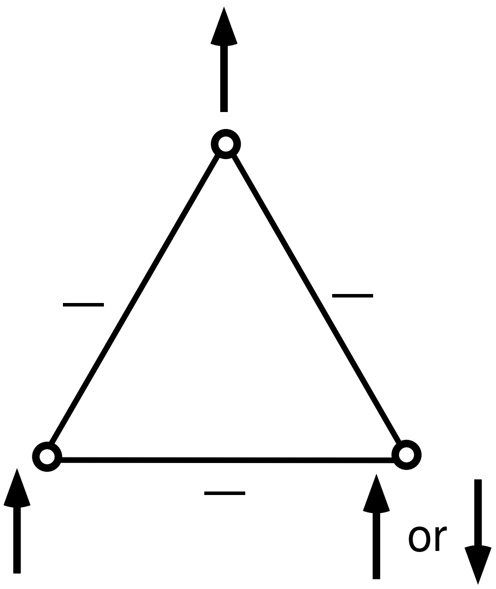

Figure 2. An illustration showing that one can have frustration without disorder. If the lattice is made up of triangles, such as a triangular lattice in two dimensions or a face-centered lattice in three dimensions, a pure antiferromagnet is frustrated.

frustrated [2]. In fact, the model in Eq. (3) with random interactions of either sign, is a well studied model (the Edwards Anderson model [3]) for spin glasses, which will be discussed later.

Note that one can have frustration without disorder [4]. For example, on lattices built out of triangles, a pure (*i.e.* non-random) antiferromagnet has frustration, as sketched in Fig. 2. We shall not, however, discuss such periodic frustrated systems here.

3. Averaging

One of the difficulties in the theory of disordered systems, is that the Hamiltonian, and hence the free energy F, depends on a *large number of parameters*, the values of all the interactions. However, we expect intuitively that the free energy of one large chunk of material will not be very different from the free energy of a different chunk with different interactions chosen from the same distribution. To see this intuitively divide up the system into many *subsystems* [5], such that each subsystem is still quite large. Then, to a good approximation, the total free energy is just the sum of the free energies of the subsystems, the error coming from the region of the boundaries, which is small compared with the volume. Since each subsystem has a different realization of the disorder, the total free energy is obtained by an average over disorder. For some simple models it can be proved rigorously that the sample to sample variations in the free energy are of order $\sqrt{N}$, where N is the number of spins and it seems likely that this is true more generally. Since the free energy itself is of order N, the relative fluctuations in the free energy vanish in the thermodynamic limit. This property is called *self averaging* and means that the *average* free energy (averaged over all configurations of the disorder) is almost the same (in the thermodynamic limit) as the free energy of a single typical sample. Hence, theoretically, we should try to calculate the average free energy (which only depends on the distribution and not on the details of an individual realization). This should then agree with experiment which is done on one large sample. To summarize, for large N we have

$$F \approx [F]_{\text{av}}, \tag{5}$$

where, F is the free energy of a single sample, the rectangular brackets $[\ldots]_{\text{av}}$ denote an average over the disorder, and the symbol $\approx$ indicates "asymptotically equal to". Note that there are two averages in the theory of random systems. The first is the the standard statistical mechanics average for a single realization of the disorder, which we denote by angular brackets, $\langle\ldots\rangle$, and the second is the average over the quenched disorder.

It is important to realize that that one should *not* average the partition function, Z. To see this, it follows from the definition

$$Z = \exp(-\beta F) \tag{6}$$

and the fact that F is extensive, (*i.e.* $F \propto N$), that $[Z]_{av}$ will be dominated by *rare* configurations of the disorder for which the free energy which is very negative. These rare configurations are of no physical interest: we are rather concerned with understanding a typical sample observed in an experiment. To make this difference concrete, assume that the distribution of free energies is Gaussian even out in the tails (it *is* Gaussian in the region with significant probability) *i.e.*

$$P(F) \propto \exp(-(F - [F]_{av})^2/2\Delta F^2), \tag{7}$$

$\Delta F(\sim \sqrt{N})$ is the width of the distribution of free energies. It is then easy to see that

$$[Z]_{av} = \int P(F) e^{-\beta F}\, dF \quad \propto \quad \exp(-\beta F^*), \tag{8}$$

where F^* is obtained by completing the square in the exponent and is given by

$$F^* = [F]_{av} - \beta \Delta F^2. \tag{9}$$

Hence F^* does not equal $[F]_{av}$ but is less than this by an amount proportional to N, see Fig. 3.

Unfortunately, averaging F is inconvenient, since the random variables are buried inside the log in the standard expression

$$[F]_{\text{av}} = -k_B T \left[\log \operatorname{Tr} \exp(-\beta \mathcal{H})\right]_{\text{av}} \; . \tag{10}$$

To get round this one often uses the "replica trick" [3, 6] in analytical work to expose the interactions in a form which can be averaged over. Since the emphasis in these lectures will be on numerical work, I will not discuss the replica trick here.

It is also possible to average correlation functions. For example, the correlation function averaged over all pairs of spins a fixed distance, r, apart is

$$C(r) = \frac{1}{N} \sum_i \langle S_i S_{i+r} \rangle, \tag{11}$$

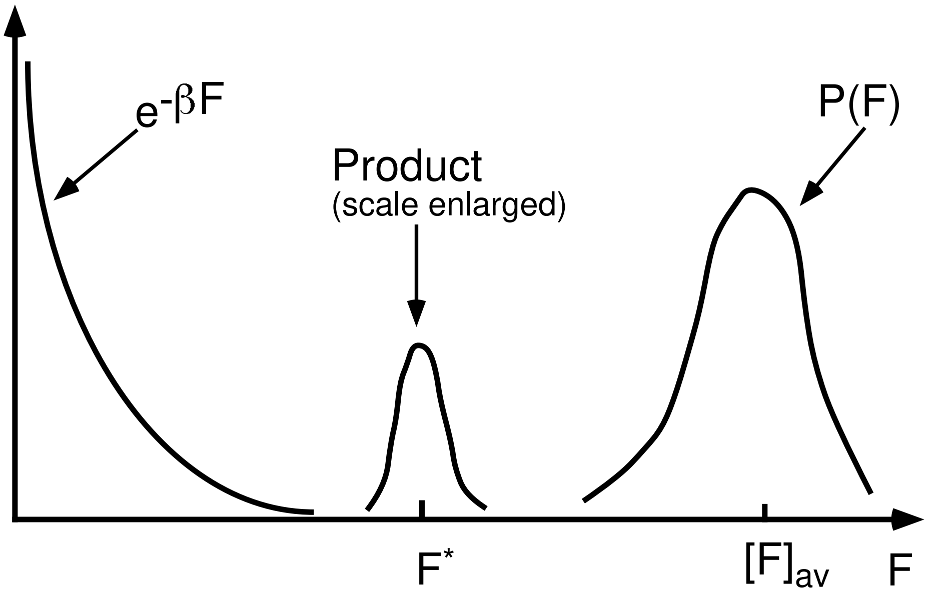

Figure 3. An illustration showing that averaging the partition function is not useful. Assuming a Gaussian distribution of free energies, of width $\Delta F(\sim \sqrt{N})$ centered on $[F]_{\rm av}(\sim N)$, the free energy corresponding to $[Z]_{\rm av}$ is F^*, given by Eq. (9), which is far out in the tail of the distribution of free energies $P(F)$. By contrast, the average free energy, $[F]_{\rm av}$ gives a result within of order $\sqrt{N}$ of the actual free energy of *any* sample, except for a small fraction which vanishes as $N \to 0$. Remember that $[F]_{\rm av}$ itself is of order N so the fractional difference between the free energy of one sample and the average free energy is of order $1/\sqrt{N}$, which *vanishes* asymptotically.

and is usually self averaging, (though in 1-d one needs very large lattice sizes to see this at large r where $C(r)$ is small). The structure factor, $S(q)$, which is the Fourier transform of $C(r)$, and which is measured in scattering experiments is self averaging in all examples that I know of.

Sample averaging is necessary not only in analytical calculations, but also in numerical work, because the sizes studied are generally too small to be self averaging. To do this average one simply repeats the calculation for many realizations of the disorder, and averages the results. Although this is simple to state, the need to repeat the calculations may times over times can demand a lot of computer time: frequently several hundred or several thousand samples are needed.

4. Random Ferromagnet

We know from the general theory of *critical phenomena* that critical behavior is characterized by various exponents which are universal, *i.e.* they only depend on general features of the system, such as the symmetry of the order parameter and the dimension of space, but not on microscopic details. One question of interest for random systems is therefore whether introduction of disorder changes the "universality class" from that of the pure system. For the case of weak disorder (*i.e.* all the interactions have the same sign and almost the same magnitude) the answer was given some time ago by Harris [7]. This argument is roughly as follows: Consider a region of (linear) size equal to the correlation length ξ. In the volume of size ξ^d, where d is the space dimension, there will be a total fluctuation of the interactions of order $\xi^{d/2}$, and a relative fluctuation of order $\xi^{-d/2}$. We can think of this as a local fluctuation of the transition temperature of order $\xi^{-d/2}$. Disorder will be irrelevant if this fluctuation in T_c is less than $T - T_c$. Using $\xi \propto (T - T_c)^{-\nu_P}$, this gives

$$\xi^{-d/2} \ll \xi^{-1/\nu_P} , \tag{12}$$

and in order that this is true if for $\xi \to \infty$ one needs

$$\nu_P > 2/d , \tag{13}$$

as the condition for weak disorder to be irrelevant. We have put the suffix P on the correlation length exponent ν to indicate it is the value for the pure system. Using the hyperscaling relation $d\nu = 2 - \alpha$, where α is the specific heat exponent, the condition for weak disorder to be irrelevant can be expressed compactly as

$$\alpha_P < 0 . \tag{14}$$

If $\alpha_P > 0$ then disorder is relevant and one expects the critical exponents of the random system to be *different* from those of the pure system. Nonetheless, one can argue that the exponents of the random system satisfy similar inequalities to those above [8], *i.e.*

$$\nu \geq 2/d , \tag{15}$$

or, if hyperscaling is valid, $\alpha < 0$, where the exponents now refer to the random system. In other words the specific heat exponent of a random system is always negative: if $\alpha_P < 0$ then $\alpha = \alpha_P$ (and the other exponents are also given by the corresponding values for the pure system) whereas if $\alpha_P > 0$ then $\alpha \neq \alpha_P$.

Let us next consider the site dilution problem which is an example of strong disorder. If the concentration of occupied sites, p, is very small, then occupied sites form small isolated clusters, see Fig. 4(a). However, as the concentration is increased, the cluster size increases until, at a critical concentration p_c, there is an infinite cluster which spans the lattice [9], see Fig. 4(b). This “percolation” problem, as it is called, is a standard second order transition with all the usual critical phenomena such as exponents. For example, the order parameter is the fraction of sites in the infinite cluster, and the quantity corresponding to the susceptibility is the average cluster size. It turns out that both the Ising model and the percolation problem belong to a family of models, called the q-state Potts model. In the Potts model, there is a variable, n_i at each site which can take one of q values, $n_i = 0, 1, \ldots, q-1$. The energy is $-J$ is neighboring sites are on the same site and 0 otherwise, so the Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{n_i, n_j} . \tag{16}$$

The Ising model clearly corresponds to $q = 2$ (apart from a shift in the energy and a redefinition of J by a factor of 2) and it was shown by Fortuin and Kastelyn [10] that the statistics of percolation clusters is described by the limit $q \to 1$. This is the same work which provides the basis for the Swendsen-Wang [11] cluster algorithm for accelerating Monte Carlo updating near a critical point (described at this school in the lectures by Gubernatis and Stauffer). The percolation problem is, in many respects the simplest phase transition, since one does not need any knowledge of quantum

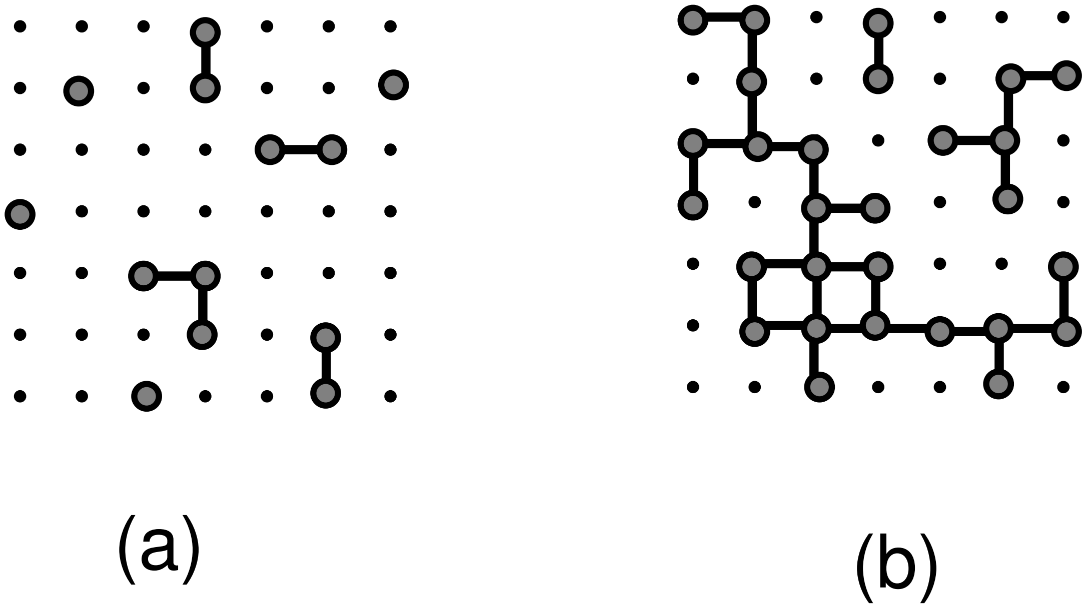

Figure 4. An illustration of the percolation problem. Each small dot represents a lattice site, and each large dot an occupied site. The sites are occupied with probability p, and a bond (represented by a line) connects nearest neighbor occupied sites. (a) corresponds to p small, for which one only has small clusters. (b) represents a value of p larger than the percolation concentration, p_c, so an "infinite" cluster has formed which spans the system.

mechanics or statistical mechanics, just the ability to join dots at the level of puzzles which are often given children. Furthermore, the physics is given directly by the structure of the clusters which is easy to visualize. Of course, to get good statistics, it is better to use the computer rather than pencil and paper, and the largest systems simulated for critical phenomena seem to have been for the percolation problem. These techniques are described in Stauffer's book [9].

Clearly one cannot have long range magnetic order in the site dilute problem if $p < p_c$, and so one expects that $T_c(p)$ decreases as p decreases, reaching zero as $p \to p_c$, see Fig. 5. At $T = 0$ all the spins in a cluster are parallel, so the magnetic properties are just given by the statistics of the clusters of the percolation problem. For example, since spins are perfectly aligned at $T = 0$, the total spin spin correlation function is just given by the mean cluster size, *i.e.*

$$T\chi \equiv \frac{1}{N} \sum_{i,j} [\langle S_i S_j \rangle]_{av} = \sum_s s^2 n_s, \tag{17}$$

where n_s is the density of percolation clusters of size s and χ is the susceptibility of the Ising system.

The percolation problem is in a different universality class from the Ising model transition at finite T because the statistics of clusters is different from the statistical mechanics of a system at finite temperature. Thus the $T = 0$ point on the phase boundary in Fig. 5 is in a different universality class from the pure Ising model at $p = 1$.

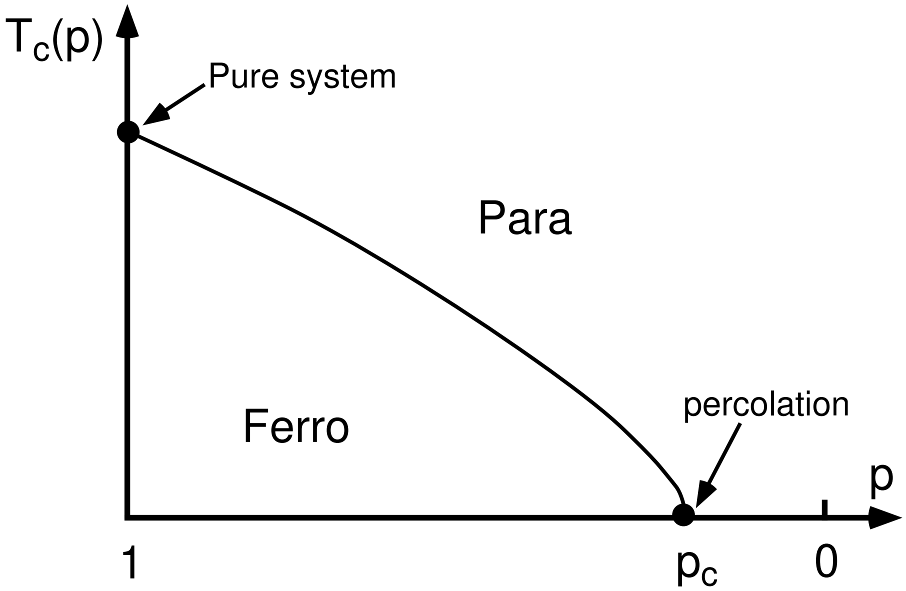

Figure 5. The phase boundary of the site dilute model in Eq. (2). As the concentration of occupied sites, p, decreases from 1, the transition temperature decreases, eventually reaching zero at $p = p_c$, the percolation concentration. The critical behavior of at the percolation point is in a different universality class from the rest of the line. If the specific heat of the pure system, α_P, is negative, then, according to the Harris criterion, weak disorder is irrelevant so the critical behavior along the line (but close to $p = 1$) should be the same as that of the pure system at $p = 1$. In fact, it seems that the critical behavior is that of the pure system *everywhere* along the phase boundary, except at the percolation point, if $\alpha_P < 0$.

What about the line connecting these two limiting cases? If $\alpha_P < 0$ then weak disorder is irrelevant, so one might expect that, at least for p close to 1, the critical behavior is that of the pure system. In practice, it seems that nothing else intervenes anywhere along the line and the critical behavior is that of the pure system everywhere except at the percolation point. This should apply to the 3-d Heisenberg model for which $\alpha_P < 0$. In principle it should also apply to the 3-d XY model but in this case α_P, while negative, is extremely small in magnitude and so the asymptotic critical behavior of the random system will only appear so close to the critical point that it will be difficult to see.

What happens if $\alpha_P > 0$? The best studied model which has this property is the 3-d Ising model. If p is close to 1, one might expect that the behavior is that of a weakly random magnet, which has been estimated, for example, by ϵ-expansion calculations [12] For arbitrary p the simplest scenario is that the system has weakly random critical behavior all the way down to (but not at) the percolation point. This is shown in Fig. 6(a). Another possibility is that different critical behavior occurs for p sufficiently small, but still greater than p_c. This is shown in Fig. 6(b).

The situation from simulations is not completely clear. It was claimed [14] that exponents are *non-universal i.e.* they vary *continuously* along the critical line. This is very strange theoretically since such behavior is rather rare, and is associated with a marginal operator in the renormalization group sense. It is not clear what the

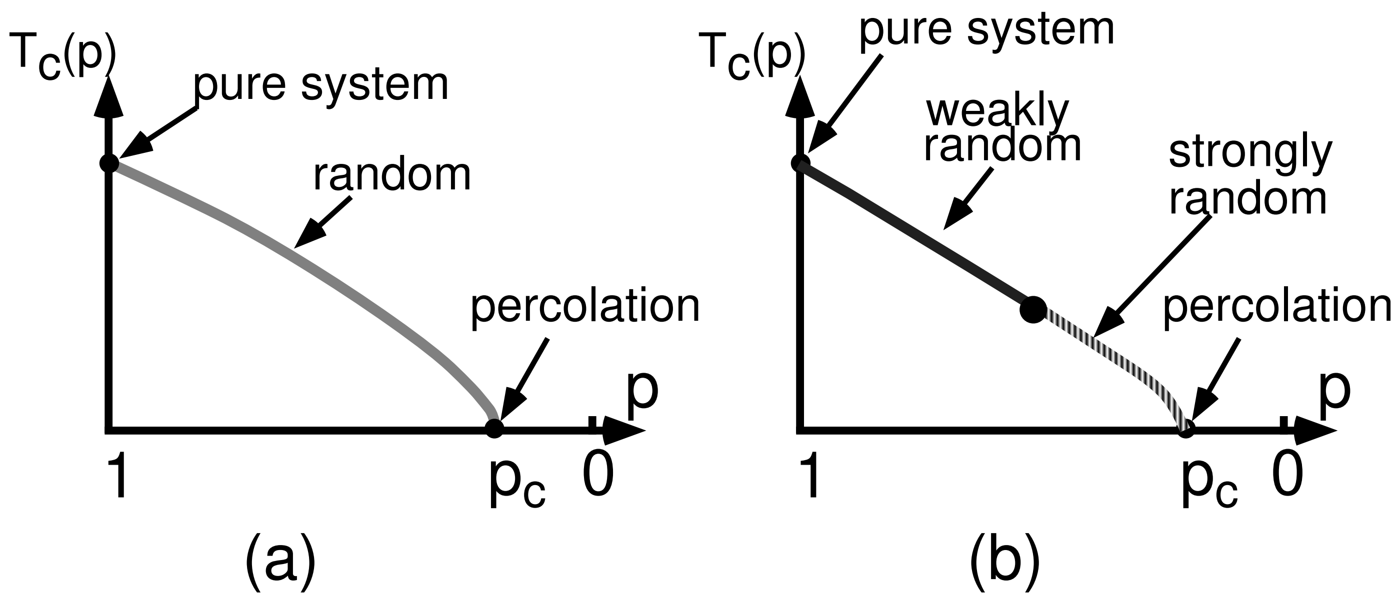

Figure 6. Two possible scenarios for the critical behavior of a site-dilute system with $\alpha_P > 0$. In (a) there are three different types of critical behavior. At $p = 1$ the system has no disorder and so has the critical behavior of the pure system. At $p = p_c$, the system has the critical behavior of the percolation problem. Everywhere else, except at these two endpoints, the system has the critical behavior of the weakly random magnet. In (b) the situation is the same except that there is a multi-critical point on the critical line, at $p = p^*$ say, separating a region of weakly random critical behavior for $1 > p > p^*$ from a region of strongly random critical behavior for $p^* > p > p_c$. It has been claimed, on the basis of numerical simulations, that neither of these possibilities occurs, but rather the critical exponents vary *continuously* along the line. In my opinion, the scenario shown in (a) is likely to be correct.

marginal operator would be here. In fact, there is *crossover* in the critical behavior at $p = p_c$ and at $p = 1$ and the apparent non-universality may be due to difficulties in separating out this crossover effect. More recently, in fact, Heuer [13] has argued that the simulation results can indeed be interpreted in this way.

The borderline case of $\alpha_P = 0$ corresponds to the most studied model in all of statistical mechanics, the 2-d Ising model, which has a logarithmic divergence of the specific heat in the pure case. There have been several calculations [15] for the weakly random case, which predict an even weaker divergence in the specific heat of the form $\log(\log(T_c/(T - T_c)))$, and the other other exponents are unchanged. There is controversy about whether numerical simulations on the dilute model show this behavior. While some authors [16] argue that the exponents are Vary continuously along the critical line, others [17] argue that a careful study shows that the behavior is universal and consistent with theoretical predictions. The most tidy answer, which is also in agreement with theoretical prejudice, is that the weakly random critical behavior occurs everywhere along the line, as shown in Fig. 6(a). Given the difficulties of separating out crossover effects at the percolation point, I do not feel that the numerics is inconsistent with this. The critical behavior of the dilute 2-d Ising model is still quite controversial and there will undoubtedly be more work done on it in the future. Since very large sizes are simulated, most work has used Swendsen Wang [11] type cluster algorithms discussed in the lectures by Stauffer and Gubernatis.

5. Random Field Ising Model

I will now discuss a problem which involves frustration, the Ising model in a random field [18, 19]. This has the deceptively simple looking Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i \ , \tag{18}$$

where the (ferromagnetic) bonds, J_{ij}, are *not* random and the h_i are random fields with zero mean and standard deviation h_r, *i.e.*

$$\begin{aligned} [h_i]_{\mathrm{av}} &= 0 \\ [h_i^2]_{\mathrm{av}} &= h_r^2. \end{aligned} \tag{19}$$

Note that this problem has frustration because there is competition between the exchange interactions which want to have the spins parallel to each other and the random fields which want the spins to locally follow the direction of the field. Although one cannot apply a random field in experiments this model is in the same universality class as the dilute antiferromagnet in a *uniform* field [20] which has been extensively studied, see Ref. [19].

A simple argument due to Imry and Ma [21] indicates that a system described by Eq. (18) should be a ferromagnetically ordered state in the presence of a weak random field for dimension greater than 2. In $d < 2$ (and a more refined argument indicates that the same holds in $d = 2$) the system breaks up into domains, in which the magnetization points in different directions in order to follow the net field in that region. To see this consider the energetics involved for the system to form a domain of size L in which the magnetization is opposite to that in the bulk. The gain in random field energy is of order $h_r L^{d/2}$ while the cost in exchange energy is proportional to the area of the wall, *i.e.* L^{d-1}. Clearly the domain wall energy is largest (at large L) if $d > 2$ so the system will not form domains and hence ferromagnetism is stable for small random fields. If $d < 2$ the random field energy is largest for large L and so the system will break up into domains (thus destroying long range ferromagnetic order) for arbitrarily weak random fields. This argument, which has been made rigorous [22], shows that a transition to a ferromagnetic state occurs in 3-d.

According to the standard theory of critical phenomena [1], there are just 2 independent critical exponents at a conventional second order transition, which can be taken to be ν (which describes how the correlation length diverges as the critical point is approached) and η (which describes the power law decay of the correlations at the critical point). For the random field problem it has been argued [23, 24, 25] that one needs 3 exponents. To see how this might arise note that one can defined two susceptibilities for the random field problem; the conventional susceptibility

$$\chi = \frac{1}{T}\frac{1}{N} \sum_{\langle i,j \rangle} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle]_{av}, \tag{20}$$

and the "disconnected" susceptibility,

$$\chi_{dis} = \frac{1}{N} \sum_{\langle i,j \rangle} [\langle S_i \rangle \langle S_j \rangle]_{av} \ . \tag{21}$$

Without the random field, χ_{dis} would vanish above T_c, but with the random field it is not zero and actually diverges more strongly than χ. To see this, consider the following simple mean field argument, in which we treat each Fourier component separately so $m_k = \chi_k h_k$, where m_k is the Fourier transform of $\langle S_i \rangle$, and h_k is the Fourier transform of h_i. Squaring and averaging over the disorder gives $\chi_{dis} \sim h_r^2 \chi^2$. Thus there is considerable cancelation between the two terms in Eq. (20): each term is more divergent than the sum. One conventionally defines two η-like exponents by

$$\chi \sim \xi^{2-\eta} \tag{22}$$

$$\chi_{dis} \sim \xi^{4-\bar{\eta}} \ . \tag{23}$$

In mean field theory $\eta = \bar{\eta} = 0$. The conventional hyperscaling relations (*i.e.* those involving the space dimensionality d) have to be modified by the replacement $d \to d - \theta$ where

$$\theta = 2 - \bar{\eta} + \eta \ , \tag{24}$$

so, for example,

$$(d - \theta)\nu = 2 - \alpha \ , \tag{25}$$

where α is the specific heat exponent. There is a rigorous inequality due to Schwartz and Soffer [26] which says that

$$\bar{\eta} \leq 2\eta \ . \tag{26}$$

Curiously, a perturbation theory calculation predicts to all orders of perturbation theory that [27, 28] $\bar{\eta} = \eta$ so $\theta = 2$, and the exponents of the random field problem are just those of the corresponding pure problem in 2 fewer space dimensions. However, this is incorrect, because it would predict no finite temperature transition in $d = 3$ (just like the 1-d pure Ising model) whereas it is known rigorously [22] that such a transition exists for weak random fields. The difficulty with the perturbation theory is that it does not adequately deal with the existence of degenerate minima of the free energy, which occurs in random systems.

Fisher [23] has argued that the exponent θ is related to properties of the (free) energy landscape. Consider, for example, a system of size L at the critical point. Fisher argues that for most samples a single minimum will dominate the statistical mechanics, but there is a probability of order $L^{-\theta}$ that there will be a second degenerate minimum. Although this possibility is rare for large L, when it does occur there are large fluctuations between the minima, so it cannot be neglected.

Schwartz et al. [29] have argued that $\bar{\eta} = 2\eta$, in which case there would again be only two independent exponents, as at conventional transitions. There is some evidence for this from series expansions [30], but it is not yet clear whether it is correct.

Both experimentally and theoretically, it has proved very difficult to study this transition in detail. A major problem, which occurs generally for frustrated systems, is that *relaxation times become very long* near the transition. There is more than one local minima in the free energy, and different minima often correspond to quite different spin configurations. Fluctuations between different minima occur by thermal fluctuation over the barrier between them, which occurs at rate of order $\exp(-\Delta F/k_B T)$, where ΔF is the height of the barrier. This is tiny when $\Delta F \gg k_B T$.

For the random field systems, one finds experimentally that no long range order is observed if one cools in a field (field cooling, FC), whereas long range order is observed if one cools in zero field and the applies the field (zero field cooling, ZFC). The different procedures are summarized in Fig. 7.

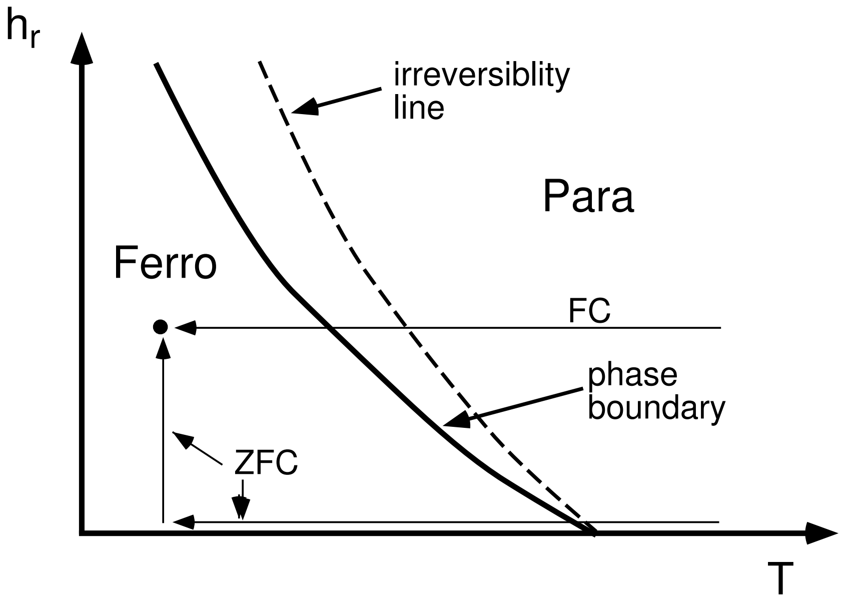

Figure 7. A sketch of the phase diagram of the Random Field Ising model. The phase boundary is shown by the thick line and the irreversibility line is indicated by the dashed line. The procedures corresponding to field cooling (FC) and zero field cooling (ZFC) are also shown.

Since we know rigorously that there is long range order (for weak random fields) the domain state found in the FC experiments must describe a non-equilibrium situation. However, it not obvious that the ZFC experiment corresponds precisely to equilibrium either, though the observation of long range order is qualitatively correct. The experiments *are* reproducible above an "irreversibility line" in the $h_r - T$ plane, which seems to lie a little above the actual transition temperature. As a result there is no general consensus, experimentally, on whether the transition is continuous or first order, or what are the values for the critical exponents.

Similar confusion exists in the numerical calculations. There have been many calculations but I will mainly describe my own work fairly recent with Heiko Rieger [32]. We used a distribution in which the random fields can only take one of two values $\pm h_r$ with equal probability. Since the energies to flip a spin are discrete, one can use multispin coding techniques [31], in which a single bit (rather than a whole word) is used to store a spin or a field. Since the word length on most computers is 32 bits it is possible to simulate at at the same time *on one machine* 32 samples with different realizations of the disorder. In addition the calculations were performed on the transputer array at Cologne, using 40 chips. Thus a total of 1280 samples were simulated in parallel. For a give lattice size the specific heat is found to have a peak at some temperature T^*, which varies with system size, see Fig. 8(a). Various quantities were evaluated at T^*. For example the peak height of the specific heat

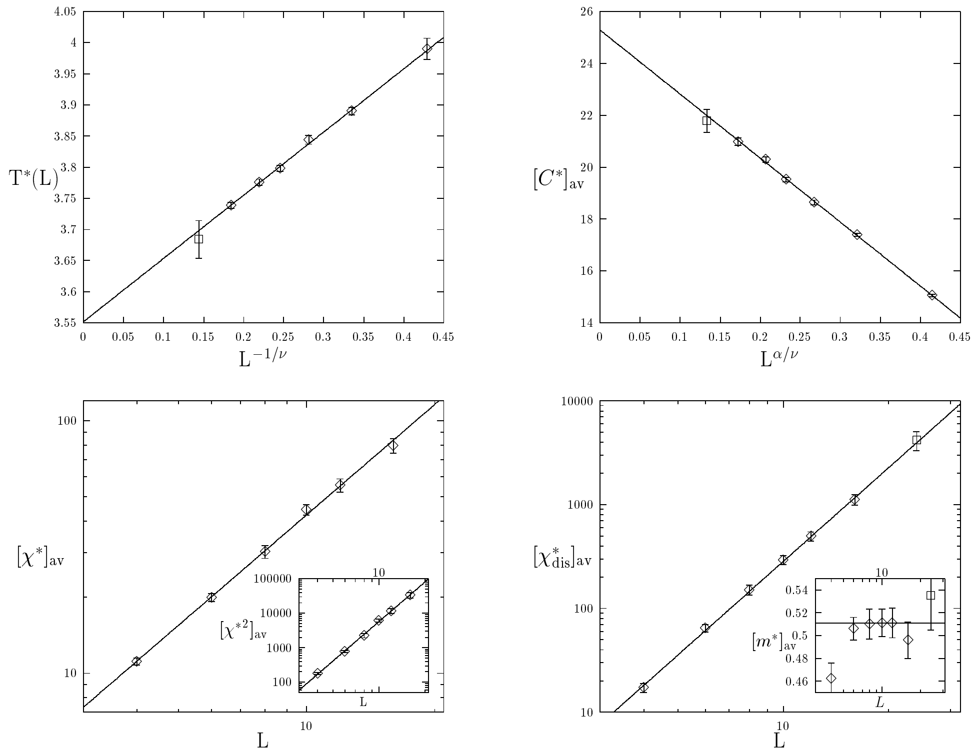

Figure 8. Data for the random field Ising model from Ref. [32] for $h_r/T = 0.35$. The points correspond to lattice sizes L=4, 6, 8, 10, 12, 16 and 24. (a) The temperature $T^*(L)$ of the specific heat maximum versus $L^{-1/\nu}$ with $\nu = 1.64$. (b) Specific heat $[C^*]_{\rm av} \equiv L^{\alpha/\nu}\tilde{C}(x^*)$ versus $L^{-1/\nu}$ with $\alpha = 1.04$ and ν as in (a). The specific heat appears to saturate for $L \to \infty$ at a value of $\lim_{L\to\infty}[C^*]_{\rm av} \simeq 25.3$. (c) Susceptibility $[\chi^*]_{\rm av} \equiv L^{2-\eta}\tilde{\chi}(x^*)$ in a log–log plot. The slope of the straight line is $2-\eta$ with $\eta = 0.53$. The inset shows the second moment $[\chi^{*2}]_{\rm av}$ of the probability distribution $P(\chi)$ in a log–log plot. The slope of the straight line is $\zeta = 3.82$. (d) Disconnected susceptibility $[\chi^*_{\rm dis}]_{\rm av} \equiv L^{4-\overline{\eta}}\tilde{\chi}_{dis}(x^*)$ in a log–log plot. The slope is $4-\overline{\eta}$ with $\overline{\eta} = 1.0$. The insert shows the magnetization $[m^*]_{\rm av} \equiv L^{-\beta/\nu}\tilde{m}(x^*)$ as a function of a L (the scale of the x–axis is logarithmic). The straight line is the extrapolation to $[m^*]_{\rm av}(L=\infty)$, which seems to be nonzero implying $\beta = 0$.

seems to extrapolate to a *finite* value as $L \to \infty$, see Fig. 8(b). On the other hand the magnetization seems to be independent of L, see the insert to Fig. 8(d). This seems to indicate that the transition may be first order in the sense of a discontinuity in the order parameter. The average susceptibility seems to diverge strongly, see Fig. 8(c). (Note that there is a very broad distribution of susceptibilities, since the average of χ^2 diverges with an exponent which is not twice that for $[\chi]_{av}$, see the insert to Fig. 8(c)).

The data for the magnetization indicates that the transition is first order, yet there is no latent heat (which would show up as a specific heat diverging as L^d), and the width of the finite-size region does not vary as L^{-d}, as expected at a first order transition, see [33]. For example, the leading correction to $T^*(L)$ varies as $L^{-1/\nu}$ with $\nu \simeq 1.64$, see Fig. 8(a). Thus these results suggest that there is jump in the order parameter but with an infinite infinite correlation length at T_c. Such behavior is unusual, but there is one system where it has already been found: the Ising model in 1-d with long range interactions which vary as $1/r^2$, which has been studied by Anderson and Yuval [34], originally in the context of the Kondo model.

There has recently been some impressive calculations by Sourlas [35] which seems to support this picture. Sourlas uses a sophisticated algorithm to calculate the exact ground state energy in a polynomial amount of time for lattices of linear size up to $L = 90$, *i.e.*$N = 90^3$, (he is therefore investigating the transition at $T = 0$ as the random field strength is varied). He finds evidence for a substantial first order jump in the magnetization, but with large fluctuations indicating that this is not a simple first order transition. Thus we see that the 3-D random field Ising model is certainly not well understood and remains a challenge for the future.

6. Spin Glasses

The last problem that I will discuss is the spin glass problem [6]. Here the disorder and frustration come entirely from the interactions. Spin glasses have been widely studied because of the universal behavior seen in a wide variety of frustrated random systems, and because ideas from spin glass theory have been applied to other areas of science such as combinatorial optimization and neural networks. A simple model, proposed by Edwards and Anderson [3], is to consider the Hamiltonian in Eq. (3) but with the interactions independent random variables with a symmetric distribution. The Gaussian distribution and the $\pm J$ distribution,

$$P(J_{ij}) = \frac{1}{2}\left(\delta(J_{ij} - J) + \delta(J_{ij} + J)\right), \tag{27}$$

are common choices. Usually one works in units with $J = 1$. There are many experimental systems for which there is disorder and frustration in the bonds [6] and for which the Edwards Anderson model should provide a reasonable description.

Because there is no net tendency to ferromagnetism or antiferromagnetism, the spin-spin correlation function $\langle S_i S_j \rangle$, will be either ferromagnetic or antiferromagnetic, in some random way depending on the realization of the bonds. Hence to see long range order it is useful to study the spin glass susceptibility,

$$\chi_{SG} = \frac{1}{N}\sum_{i,j}[\langle S_i S_j \rangle^2]_{av}, \tag{28}$$

which will diverge if the range over which the spins are correlated becomes large, irrespective of whether that correlation is ferromagnetic or antiferromagnetic. Although this cannot be directly measured experimentally a closely related quantity, the non-linear susceptibility, χ_{nl} *can* be measured. χ_{nl} is defined in terms of the expansion of the magnetization m in powers of an applied uniform field h, by

$$m = \chi h - \chi_{nl} h^3 + \cdots. \tag{29}$$

From statistical mechanics it follows that χ_{nl} is a 4-spin cumulant average. For the Edwards Anderson model, the only piece of this which survives averaging over the disorder is the square of the 2-spin correlation function, so

$$\chi_{nl} \propto \chi_{SG} \ . \tag{30}$$

There is clear experimental evidence [36] that χ_{nl} becomes huge and probably diverges, indicating large collective effects and probably a finite temperature transition.

A goal of the theory, which I will discuss briefly here, is to see whether a transition really occurs and to ascertain the values of universal critical exponents. As we shall see, the situation is not completely clearcut despite considerable computational effort. Other aspects, which are perhaps even less well understood and which I will not discuss for lack of time, are the nature of the ordered phase, and non-equilibrium effects. The latter is very important because the system is never fully in equilibrium in the low temperature phase.

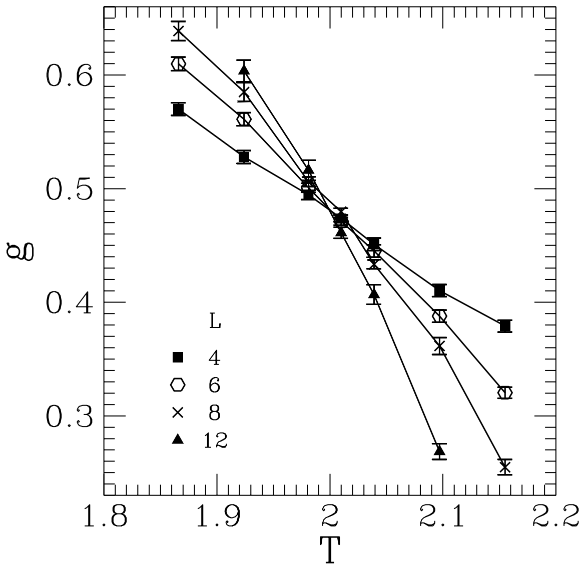

Figure 9. Data for g for the 4-dimensional Ising spin glass with a $\pm J$ distribution. The crossing of the curves for different sizes shows clearly that there is a finite T_c with non-zero spin glass order below T_c.

Several studies have been carried to determine whether or not there is a transition [37, 38, 39, 40, 41, 42] I will now briefly describe some of the Monte Carlo simulations that I carried out together with Ravin Bhatt [41] about 10 years ago, and very recently with Naoki Kawashima [42]. Since we need to perform the square of the thermal average in Eq. (28) it is convenient to simulate two copies of the system with identical bonds. We calculate the "overlap" between the spins in the two copies *i.e.*

$$Q = \frac{1}{N} \sum_i S_i^{(1)} S_i^{(2)} \ . \tag{31}$$

The expectation value of Q is just the spin glass order parameter, q, defined by Edwards and Anderson

$$q = [\langle S_i \rangle^2]_{av} = [\langle Q \rangle]_{av}, \tag{32}$$

and the average of Q^2 is related to the spin glass susceptibility

$$\chi_{SG} = N[\langle Q^2 \rangle]_{av} \ . \tag{33}$$

It is also interesting to look at other moments of Q, particularly the dimensionless Binder ratio [43]

$$g = \frac{1}{2}\left\{3 - \frac{[\langle Q^4\rangle]_{av}}{[\langle Q^2\rangle]_{av}^2}\right\} , \tag{34}$$

normalized so that $0 \le g \le 1$. As discussed in the lectures by Dunweg, see also [33], the finite size scaling form for this is particularly simple

$$g = \tilde{g}(L^{1/\nu}(T - T_c) , \tag{35}$$

so curves for g against T for different sizes should come together at T_c and splay out again in the opposite sense below T_c. This is useful because T_c can be determined *independent* of the other exponents.

Because the approach to equilibration is rather slow, one has to be very careful to ensure that enough Monte Carlo sweeps were carried out, for each size and temperature, for the system to be in equilibration. Techniques for doing this, which seem to work out well in practice are described in ref. [41]. Fig. 9 shows some previously unpublished results for $d = 4$ with a $\pm J$ distribution of bonds, in which provide clear evidence of a finite transition temperature. The value of T_c can be read off from the point of intersection and the exponent ν can be obtained by collapsing all the data in the form expected from Eq. (35). In the old data for $d = 3$ (for a $\pm J$ distribution) [40, 41] the curves come together at about $T = 1.2$ but then, within the error bars stick together at lower temperatures. This indicates near marginal behavior, or, in other words, the lower critical behavior of the Ising spin glass is close to 3. From this data one cannot tell whether there is a finite T_c with ordering below T_c (corresponding to a splaying out below T_c not seen because of the sizes are too small and/or the statistics are not good enough), whether $T_c = 0$ but with an exponential divergence at $T = 0$ (this possibility has recently been emphasized by the Rome group [44], or whether there is a line of critical points, terminating at a finite T_c as happens in the Kosterlitz-Thouless theory of the 2-d XY model.

Although the calculation with Bhatt [40, 41] was state of the art when it was carried out a decade ago, (parallel processing on the Distributed Array Processor), computers have become so much more powerful in the meantime that it seemed worthwhile to look at the problem again. Unfortunately there is currently no cluster algorithm for frustrated systems, analogous to the Swendsen-Wang algorithm for unfrustrated systems [11], so spin glass simulations require considerable computer time. In recent work with Kawashima [42] we used the multispin coding techniques, mentioned above, for the $\pm J$ model running on many processors of the SP2 at the Maui High Performance Computing Center. (Since the parallelism is trivial, one just runs different realizations of the disorder on different machines so no communication between machines is needed, we used the SP2 as a bunch of independent IBM 390 workstations). We are able to study larger sizes in the vicinity of the possible T_c and obtain much better statistics (error typically less than 0.002) because we averaged over a larger number of samples, at least 8000 for $L \le 12$, at least 4000 for $L = 16$ and at least 2000 for $L = 24$. The data shown in Fig. 10 intersects at about 1.1 (a little lower than the earlier estimate), but now is seen to splay out, though only slightly, at lower T. The data scales well assuming $T_c = 1.1$, and a fairly large value for ν of about 2, see Fig. 11. Since the splaying out is small, about 2–3 standard deviations for 3 or 4 points, one might worry that this either a fluke of the the statistics, or a small systematic error possibly caused by corrections to finite size scaling. Nonetheless, the

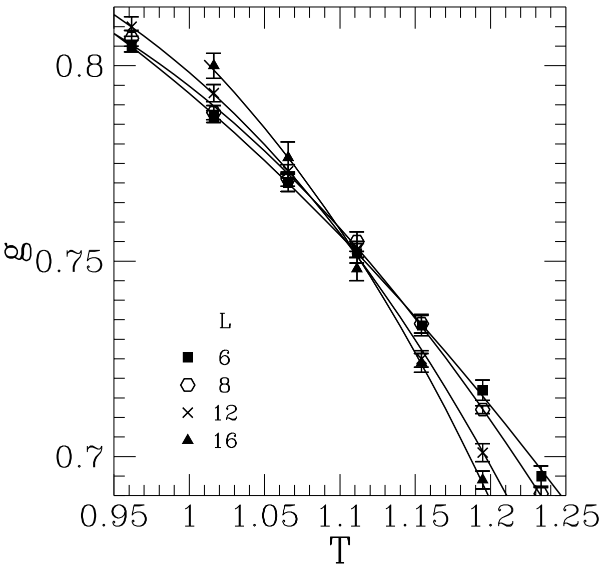

Figure 10. Results for the Binder ratio g for the 3-d $\pm J$ Ising spin glass, for different sizes and temperatures from Ref. [42]. The lines are smooth curves through the data and are only intended as guides to the eye.

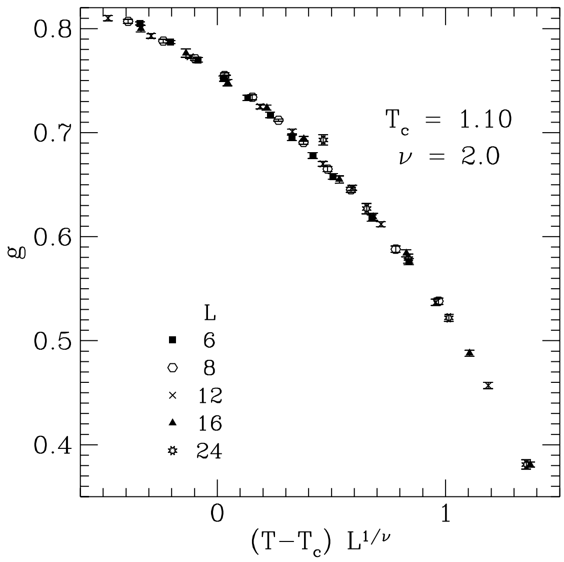

Figure 11. A scaling plot for the data in Fig. 10 from Ref. [42].

most plausible explanation of the data is that there *is* a finite T_c but the possibilities that $T_c = 0$ or that there is Kosterlitz-Thouless behavior still cannot be completely ruled out.

7. Conclusions

There remain some major unsolved problems in the study of phase transitions in random systems, which are challenges for the students at this school. Among these are:

(i) For the dilute ferromagnet, do the exponents vary continuously along the critical line?

(ii) For the random field Ising model, is the magnetization continuous or not at the transition? If continuous, what are the critical exponents?

(iii) For the spin glass:

   (a) can one show *beyond reasonable doubt* that there is a finite temperature phase transition?
   (b) what is the nature of the ordered phase [45]?
   (c) can one understand non-equilibrium phenomena, such as aging [46]?
   (d) can one understand *quantum* spin glasses [47], in which rare strongly correlated regions give rise to important Griffiths singularities?

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