Quantum Phase Transitions

A. P. Young[a] *

[a]Physics Department, University of California, Santa Cruz, CA 95064

In this talk I will try to summarize current activity in the condensed matter physics community on quantum phase transitions with disorder. I will describe some examples and explain in what ways these transitions differ from conventional phase transitions at finite temperature. In particular, similarities with, and differences from, phase transitions studied in lattice gauge theories will be brought out.

1. INTRODUCTION

Lattice gauge theorists usually study what I am calling here "quantum" phase transitions, *i.e.* phase transitions driven by quantum, rather than thermal, fluctuations. In condensed matter physics, though, such transitions have received less attention than the more conventional "thermal" phase transitions where one goes through the transition by varying the temperature. However, there has recently been an upsurge of interest in quantum phase transitions in the condensed matter community, partly because there are experimental systems, and partly because, *in the presence of quenched disorder*, such transitions exhibit new physics which is rather poorly understood.

Let us first discuss why one can generally set $\hbar = 0$ in treatments of phase transitions in condensed matter physics. The thermal energy per degree of freedom is $k_B T$, where k_B is Boltzmann's constant, while the zero-point quantum of energy is $\hbar\omega_c$, for a fluctuation of characteristic frequency ω_c. As one approaches a phase transition, the fluctuations in space become correlated over a distance, ξ, which diverges as the critical point is approached. Denoting by δ the deviation from criticality (δ would be $\propto T - T_c$ for a thermal phase transition at temperature T_c), the correlation length generally diverges with a power law, *i.e.*

$$\xi \sim \delta^{\nu} \ , \tag{1}$$

which defines the correlation length exponent, ν. As the critical point is approached, not only do the fluctuations extend over large spatial scales, they also become very slow, and there is a divergent relaxation time τ. This effect is known as critical slowing down. It is conventional [1] to define a dynamical exponent, z, by

$$\tau \sim \xi^{z} \ . \tag{2}$$

The appropriate quantum of energy is then $\hbar\omega_c$ with $\omega_c \sim 1/\tau$. Hence, if the transition is at a finite temperature, T_c, quantum mechanics is irrelevant for critical phenomena, *i.e.* $\hbar\omega_c \ll k_B T$, provided

$$\delta \ll T_c^{1/z} \ , \tag{3}$$

which is *always* satisfied sufficiently close to T_c, *provided T_c is finite*. In this regime one observes the critical behavior of the classical (thermal) transition.

Note, however, that the lower is T_c, the closer to criticality one has to be to observe classical critical behavior. If T_c is forced to zero by changing some parameter in the Hamiltonian, and the transition at $T = 0$ is traversed by changing this parameter, then one would *never* be in the classical regime. Instead, one would observe critical behavior of a *quantum universality class* which is different, in general from the *classical universality class* of the finite T_c transition. Fig. 1 sketches a typical phase diagram [2], in which Δ denotes the parameter which drives T_c to zero. One sees that the regime of classical critical behavior shrinks to zero as $T_c \to 0$. The new quantum universality class is observed in the quantum disordered

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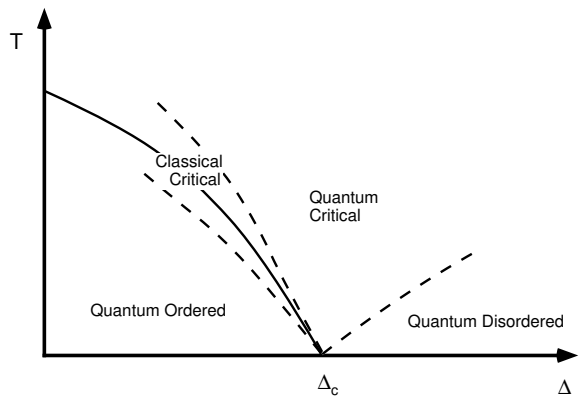

Figure 1. A sketch of the various regions which occur when a parameter Δ drives T_c to zero. The phase boundary is indicated by the solid line. The dashed lines indicate regions where crossovers occur between different types of behavior. The region of classical critical behavior shrinks to zero as $T_c \to 0$. The quantum universality class is obtained as one goes through the quantum critical point at $T = 0, \Delta = \Delta_c$ from the quantum disordered to the quantum ordered regime.

and quantum ordered regimes. In the intermediate regime (sometimes called the quantum critical regime) the only energy scale is $k_B T$ and the there are interesting predictions [3] for the dynamical scaling functions which turn out to depend only on the ratio $\hbar\omega/k_B T$. I shall not, however, discuss this regime here.

It is well known [4] that the quantum problem can be described in terms of an effective classical action in d space plus one imaginary time dimensions. The size of the system in the imaginary time direction is $\hbar/k_B T$. In this picture it is easy to see why quantum mechanics is only important at $T = 0$. At any finite temperature we are dealing with a slab, infinite in the space directions but finite in the time direction. When the correlation "length" in the time direction reaches the thickness of the slab, the growth of correlations with time becomes limited by the slab thickness, and can be described by the techniques of finite size scaling [5]. The critical behavior is then just controlled by the diverging correlation length in the space directions, *i.e.* we have a d-dimensional classical problem. However, when T is zero, the size of the system in the time direction is infinite, just as it is in the space directions, so we have to study a d+1 dimensional problem, which will generally have different critical behavior from that of the d-dimensional model. Thus the quantum to classical crossover is similar to a dimensional crossover.

Note that for systems with quenched disorder, the disorder is independent of (imaginary) time and so is *perfectly correlated* in the time direction. Although condensed matter physicists have dealt with disordered systems for some time, there is not much intuition for what happens when the disorder is strongly correlated.

In simple cases, and with no disorder, one ends up with a classical model in one higher dimension [6], in which space and time are equivalent. Another way of saying this is that the dynamical exponent, z, is equal to one. This is the situation in lattice gauge theories, because the theory should be Lorentz invariant in the continuum limit. However, condensed matter physics is nonrelativistic so space and time do not have to be equivalent. Indeed, in the cases of greatest interest, *i.e.* those with quenched disorder, space and time are *not* equivalent, and $z \neq 1$ in general. This complicates the analysis of numerical data because scaling is *anisotropic*. Analysis of numerical data then has to be done using anisotropic finite-size scaling, see *e.g.* [7,8].

Thus, there are some *similarities* between quantum phase transitions in condensed matter physics and lattice gauge theories because in both cases:

1. one studies d+1 dimensional models,
2. one frequently uses Monte Carlo simulations to do calculations,
3. the numerical data is analyzed by finite size scaling.

There are, however, also a number of important *differences* between lattice gauge systems

and quantum, condensed matter systems with quenched disorder (which we will concentrate on in the rest of this talk) because, the condensed matter systems have the additional features that:

1. there is quenched (*i.e.* time independent) disorder.
2. as a result, space and time are inequivalent, so $z \neq 1$ in general.
3. the finite size scaling analysis of the data is more complicated because scaling is anisotropic.

In the rest of this talk I will give some examples of quantum phase transitions with disorder in condensed matter physics, discuss what new physics arises in these systems, and summarize why this is rather poorly understood. I will not go into details on the numerical calculations: information on these can be found in the references.

2. EXAMPLES

The most commonly studied quantum phase transition is the metal-insulator transition [9]. The basic ideas were laid down by Anderson many years ago and are as follows: Consider a system of non-interacting electrons. In a translationally invariant system, we know that all eigenstates are *extended*, *i.e.* they have an amplitude of order $1/\sqrt{N}$ on all sites (where N is the number of sites). Electrons in such states can conduct. However, in a disordered system, some or all of the states are *localized*, *i.e.* they have a finite amplitude to be on a site in a particular region of the lattice, and negligible amplitude to be outside this region. Electrons in localized states cannot conduct electricity across the sample at $T = 0$ so, if the Fermi energy lies in a region of localized states, the system is an insulator. (At finite temperature however, conduction can occur because *inelastic* scattering processes, *e.g.* due to phonons, can kick an electron from one localized state to another.) In three and more dimensions, and for weak disorder, only the states at the edge of the band are localized, and there is a "mobility edge" separating these states from extended states at the center. If the degree of disorder, or the density of electrons, is changed, the Fermi energy can go from being in a region of extended states to a region of localized states: this is the metal-insulator transition. At a critical value of disorder, *all* the states become localized. It turns out that in two dimensions, all states are localized for *any* non-zero disorder, *i.e.* the critical value of disorder is zero.

Denoting by δ the deviation from criticality then the $T = 0$ conductivity is found to vanish on the metallic side like

$$\sigma \sim \delta^{\mu} \ , \tag{4}$$

and, of course, to be identically zero on the insulating side. One goal of theory is to calculate the exponent μ and compare with experiment. Numerical work [10] finds $\mu \simeq 1.4$ and the experimental values [10] range from about this down to much lower values. This wide range reflects the difficulty of extrapolating the data to $T = 0$ and possibly also indicates a rather narrow critical region (the region in which asymptotic scaling behavior is seen).

Most theoretical work neglects interactions and studies the Anderson Hamiltonian,

$$\mathcal{H} = \sum_i \epsilon_i c_i^\dagger c_i - t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) \ , \tag{5}$$

where the c and $c^\dagger$ are destruction and creation operators for electrons, and the ϵ_i are random site energies chosen from some distribution.

Analytical work has been developed for this model, based on a $2 + \epsilon$ expansion [11], but this does not seem to converge well in $d = 3$. Unfortunately, there is no expansion away from an upper critical dimension (which, in the case of classical critical phenomena, is better behaved), and it is not known whether an upper critical dimension (above which the exponents would stick at some "mean field" value) even exists.

One difficulty with the metal-insulator transition is that there is no obvious order parameter. This is not the case for my second example, the superconductor-insulator transition in disordered thin films. I have already mentioned that there is no metallic state in a two-dimensional system with disorder because all the electron states are

localized. However, superconductivity (which involves a coherent propagation of a *pair* of electrons) is not ruled out, though it too will presumably be eliminated if the disorder is strong enough. One has therefore the possibility of a direct transition from a superconductor to an insulator without going through an intervening metallic phase, as one changes the degree of disorder. The order parameter is just the amplitude of the pair condensate, as usual in a superconducting transition, and so one can formulate standard methods, such as Ginzburg-Landau theory.

An interesting prediction [12] is that the system *is* metallic (*i.e.* has a finite conductivity) precisely *at* the critical point and that the value of the conductivity there, σ_c, is *universal*. To be more precise,

$$\sigma_c = C\sigma_Q \ , \tag{6}$$

where C is a universal constant and

$$\sigma_Q = \frac{(e^\star)^2}{h} \tag{7}$$

is the quantum of conductance, with $e^\star = 2e$, the charge of a Cooper pair. A goal of the theory is to calculate this universal conductance as well as standard universal quantities such as critical exponents. Experimental estimates [13] of σ_c vary over a range of about a factor of three, which, if correct, would mean that σ_c is not universal after all, but it is possible that many of the experiments did not probe the asymptotic critical region, often very small in superconductors.

The simplest theoretical approach is to assume that the transition is driven by bose fluctuations, and to treat the Cooper pairs as bosons. An appropriate model is then the boson Hubbard model [14]:

$$\mathcal{H} = \sum_i \epsilon_i \hat{n}_i + U \sum_i \hat{n}_i(\hat{n}_i - 1)$$

$$-t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + c_j^\dagger c_i) \ , \tag{8}$$

where $\hat{n}_i$ is the boson occupation number on site-i. Note that, unlike the electron case, it is essential to include an on-site repulsion, U, otherwise *all* the bosons would go into the lowest localized state.

Although this model can be reformulated in a Ginzburg-Landau picture, an ϵ-expansion about an upper critical dimension does not work [14], since the renormalization group flows go to strong coupling. In fact, as we shall see, it is possible that there is actually no upper critical dimension. Since analytic approaches don't work, much of what we know has been learned from Monte Carlo simulations, *e.g.* [7].

The third and final example of a quantum phase transition which I will discuss is a magnetic system. Suppose we have an Ising magnet, where the spins lie up or down. These are just eigenstates of the Pauli spin matrix σ^z. The Hamiltonian can therefore be written

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z \ . \tag{9}$$

We shall be interested in the case where the interactions J_{ij} are random and *frustrated*, *i.e.* have random signs so that no spin configuration simultaneously minimizes all the terms in the Hamiltonian. Such a system is called a *spin glass* [15]. Typically one chooses an Edwards-Anderson model, considering only nearest neighbor interactions which are independent random variables chosen from a symmetric distribution, *i.e.*

$$[J_{ij}]_{\rm av} = 0 \ ; \quad [J_{ij}^2]_{\rm av} = J^2 \ . \tag{10}$$

So far, all operators commute so one could just replace the spin operators by c-numbers, ± 1, which the usual way to present the Ising model. Next, however, add a transverse field which couples to σ^x so the Hamiltonian becomes

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{i,j} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x \ . \tag{11}$$

The problem is now quantum mechanical since σ^x and σ^z don't commute. If the system has a finite-temperature transition with $\Gamma = 0$, then turning on Γ will decrease T_c since the transverse field cants the spins away from their ordering direction (the z-direction). If Γ is big enough, T_c can be depressed to zero and a quantum critical point results.

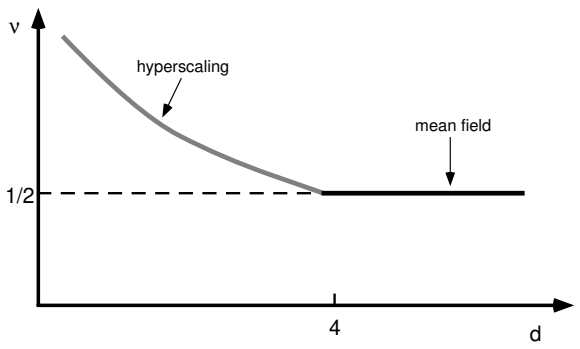

Figure 2. A sketch of the dependence of the correlation length exponent, ν, as a function of dimensionality, d, for a thermal transition. Above the upper critical dimension $d_u(=4)$, ν sticks at its mean field value of 1/2. As $d \to d_u^-$ the value of ν approaches its mean field value continuously permitting an expansion in powers of $\epsilon \equiv d_u - d$.

A quantum spin glass system for which Eq. (11) should be a good model has been studied quite recently in Ref. [16]. One advantage of this system is that one can tune through the quantum transition by varying the transverse field, a continuous parameter, whereas in the other systems I have mentioned one generally has to vary the degree of disorder, which cannot be changed continuously and for each data point one has to grow a new sample.

Again, for the quantum spin glass, we do not have an ϵ-expansion away from an upper critical dimension, so the only estimates for critical exponents, at least for short range interactions, have come from Monte Carlo simulations [8,17] on the model described by Eq. (11).

3. ABSENCE OF AN UPPER CRITICAL DIMENSION?

One of the most successful theoretical approaches to study transitions driven by thermal fluctuations has been the implementation of the renormalization group using an expansion in terms of $\epsilon \equiv d_u - d$, where d_u is the upper critical dimension, generally equal to 4. For $d > d_u$ critical exponents stick at their mean field values, and, as $d \to d_u^-$, exponents approach their mean field values continuously, see Fig. 2. Furthermore, for $d \leq d_u$ hyperscaling [18] relations hold between exponents. These relations, such as

$$d\nu = 2 - \alpha \tag{12}$$

where α is the specific heat exponent, involve the space dimension d. Clearly they cannot hold for $d > d_u$ where the exponents are independent of d, but violation of hyperscaling in this region is quite easy to understand in terms of a "dangerous irrelevant" variable [18]. Only at $d = d_u$ does one have hyperscaling satisfied with mean field exponents. Thus, if one knows the mean field exponents one can deduce d_u. For example, at a thermally driven transition the mean field values are

$$\nu_{mf} = \frac{1}{2} \; ; \quad \alpha_{mf} = 0 \; , \tag{13}$$

and at d_u we must have

$$d_u \nu_{mf} = 2 - \alpha_{mf} \; , \tag{14}$$

which gives the expected result, $d_u = 4$.

Next let us discuss how the hyperscaling relations are modified for a quantum phase transition. The appearance of the dimension, d, arises from consideration of fluctuations in a correlation volume, ξ^d for a thermal transition. In the quantum case, we also have to include the correlations in the time direction, so the analogous quantity is $\xi^d \tau \sim \xi^{d+z}$. Thus d gets replaced in the hyperscaling relations, not by $d+1$ as one might have first guessed, but by $d+z$. As an example, Eq. (12) becomes

$$(d+z)\nu = 2 - \alpha \; . \tag{15}$$

In many cases, the mean field values of α and ν are the same as in the classical case. (This is true for the superconductor-insulator transition: the quantum spin glass is more complicated, see Ref [19].) Assuming then that we can still use $\alpha_{mf} = 0, \nu_{mf} = 1/2$, Eq. (15) gives

$$d_u + z_{mf} = 4 \; . \tag{16}$$

Now the mean field value of z is equal to 2, and, more generally, one expects that $z > 0$ since the fluctuations surely slow down, rather than speed up, as the critical point is approached, so we have the inequality

$$d_u < 4 \ . \tag{17}$$

For random systems the following inequality, involving the correlation length has been established [20]

$$\nu \geq \frac{2}{d} \ , \tag{18}$$

where d is the number of dimensions in which the system is disordered. In our case, there is no disorder in the time direction, so d is just the space dimensionality, as implied by the notation. Roughly speaking, Eq. (18) is obtained by requiring that the critical behavior is unchanged if a small *additional* amount of disorder is added. For thermal transitions where conventional hyperscaling, *e.g.* Eq. (12), holds, Eq. (18) becomes $\alpha \leq 0$, which is known as the Harris [21] criterion. Applying Eq. (18) at the upper critical dimension, where we can use the mean field value of ν, we get

$$d_u \leq 4 \ , \tag{19}$$

which is inconsistent with Eq. (17). We thus conclude that in many cases of quantum transitions with disorder one cannot have an upper critical dimension at which the exponent values are continuous *i.e.* the exponents do not have an expansion in $\epsilon \equiv d_u - d$. Two scenarios seem possible: either there is no upper critical dimension, the exponents vary smoothly with d up to $d = \infty$ without saturating at some finite d, or there is an upper critical dimension, whose value could be non-universal, at which the exponents jump discontinuously.

4. GRIFFITHS SINGULARITIES

In this section I will discuss an aspect of phase transitions in disordered systems which has no counterpart in pure systems. These effects, called Griffiths singularities [22], are rather weak at thermal phase transitions but can give surprising and spectacular effects in the quantum case.

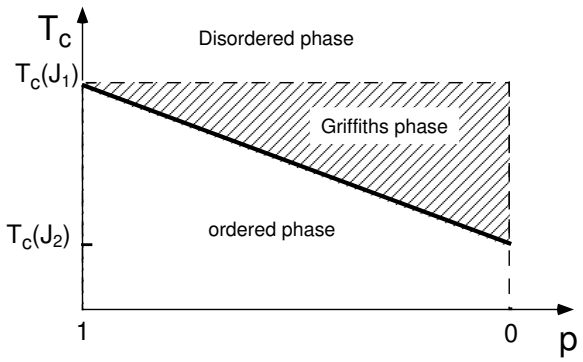

Figure 3. A sketch of the phase diagram of the model described by Eqs. (9) and (20), showing the Griffiths phase, where the free energy is a non-analytic function of magnetic field h. For this classical model the singularity is very weak, an essential singularity. However, the corresponding singularity is much stronger for quantum transitions at $T = 0$, see Fig. 4.

To illustrate Griffiths singularities, consider first a simple *classical* spin model, as in Eq. (9), where the interaction can take one of two values, *i.e.*

$$J_{ij} = \begin{cases} J_1 & \text{prob. } p \\ J_2 & \text{prob. } 1-p \end{cases} \tag{20}$$

Assume that J_1 and J_2 are both positive, so there is no frustration, and $J_1 > J_2$. The resulting phase diagram is sketched in Fig. 3. The Griffiths phase is the region which would be below T_c if all the interactions had their largest possible value (J_1 in this case) but is actually above the T_c of the random system. Griffiths singularities arise in the Griffiths phase because there is a probability of having a large region with all J_1 bonds which would be locally ordered. Since an ordered system has a large response to a magnetic field, h, the free energy turns out not to be an analytic function of h at $h = 0$ in the Griffiths phase [22]. The probability of having a large region with all J_1 bonds is exponentially small, and, as a result, the singularity is very weak at a ther-

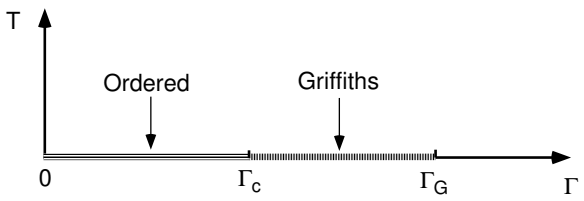

Figure 4. A sketch of the phase diagram of the one dimensional random transverse Ising model. The ordered phase only occurs at $T = 0$. In the Griffiths phase the free energy is a non-analytic function of magnetic field h, the singular part varying as in Eq. (22), where δ is a function of Γ tending to unity as $\Gamma \to \Gamma_c^-$ and tending to infinity as $\Gamma \to \Gamma_G$.

mal transition, just an essential singularity [23]: all derivatives of the free energy with respect to h are finite.

Although the effects of Griffiths singularities on static properties are probably unobservably weak for thermal transitions, they do have a bigger effect on dynamics, at least for Ising spins, because spins which are locally ordered need to go over a large barrier, ΔE, to reorient, which takes an exponentially large time of order $\exp(\Delta E/k_B T)$. Randeria et al. [24] showed that, as a result, the local spin-spin correlation function cannot decay faster than

$$\langle \sigma_i(t)\sigma_i(0)\rangle \sim \exp\left[-c\,(\log t)^{\frac{d}{d-1}}\right] \ , \tag{21}$$

which is slower than an exponential but faster than a power. Although, to our knowledge, this precise form has not been convincingly seen in simulations, slow dynamics *is* observed [25] in random systems below the critical temperature of the pure system.

Now consider what effects Griffiths singularities have at quantum transitions. Since statics and dynamics are mixed, we might expect a fairly big effect also in statics and this turns out to be correct, as we shall see.

Griffiths singularities have been worked out in great detail for one quantum model: the one-dimensional version of Eq. (11). This was first studied by McCoy and Wu [26] in the equivalent (1+1)-dimensional classical version, an Ising model with disorder which is perfectly correlated along one direction. Subsequently, new results and a great deal of insight were obtained by Fisher [27]. A one-dimensional system with short range interactions has no long range order except at $T = 0$, so the ordered region is entirely along the Γ axis, as shown in Fig. 4. The system is ordered up to a critical value Γ_c. However, there is a finite region in the disordered phase, $\Gamma > \Gamma_c$ where the susceptibility is infinite. This is most unusual because the susceptibility normally diverges *at* the critical point. In fact the free energy is a non-analytic function of field, h, in the disordered region, up to $\Gamma = \Gamma_G$, the end of the Griffiths phase. Γ_G is the value Γ_c would have if all the interactions had their largest value. Unlike the essential singularity in the case of thermal transitions, one has a much stronger effect here:

$$f_{sing} \sim |h|^{\delta(\Gamma)} \ , \tag{22}$$

where δ is a function of Γ tending to unity as $\Gamma \to \Gamma_c^-$ and tending to infinity as $\Gamma \to \Gamma_G$. The value of Γ at which the susceptibility diverges is now seen to have no special significance, but is just the point where $\delta = 2$.

These results occur because of rare regions which are more strongly correlated than average and are locally in the ordered phase. A rough argument, which applies for arbitrary d, goes as follows: The probability of finding a region of volume L^d which is locally in the ordered phase is of order

$$P(L) \sim e^{-\mu L^d} \ , \tag{23}$$

where μ is some constant. Consider now the spins as a function of (imaginary) time in this region. To change the sign of the magnetization requires inserting a domain wall, whose energy is σL^d, where σ is the surface tension. Hence the relaxation time in this region is of order

$$\tau \sim e^{\sigma L^d} \ . \tag{24}$$

In other words, there is an exponentially small probability of having an exponentially large re-

laxation time. Consequently the distribution of relaxation times follows a power law,

$$P(\tau) \sim \frac{1}{\tau^{1+\frac{\mu}{\sigma}}} \ , \tag{25}$$

as does the local time dependent spin-spin correlation function,

$$\langle \sigma_i^z(t)\sigma_i^z(0)\rangle \sim \frac{1}{t^{\frac{\mu}{\sigma}}} \ , \tag{26}$$

where t is imaginary time. The local susceptibility is obtained by integrating over t and so diverges if $\mu/\sigma < 1$.

This argument, while crude, suggests that the form of the singularity in Eq. (22) should apply quite generally to the Griffiths phase at $T = 0$ in Ising systems, though the dependence of δ on Γ presumably depends on the model. In particular, it is not clear whether the susceptibility will diverge in the Griffiths phase except in $d = 1$.

5. CONCLUSIONS

We have seen that while there are aspects in common between lattice gauge theories and models of condensed matter systems undergoing quantum phase transitions, there are also a number of differences, particularly for problems with quenched disorder. The physics of quantum phase transitions with disorder remains poorly understood compared with our rather complete understanding of thermal phase transitions. We have seen that non-perturbative effects (Griffiths singularities) have a big effect, and it is in interesting to speculate whether there may be a connection between non-perturbative effects and the lack of an ϵ-expansion.

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