# Bosons in a Random Potential

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These lectures discuss a quantum transition at zero temperature between the Bose-glass and superfluid phase of bosons moving in a random potential in two-dimensions. This problem illustrates the methods used to study quantum transitions *in general*, and the difficulties which arise when the system is disordered. This problem should also describe the observed superconductor-insulator transition in disordered thin films. After introducing a suitable model Hamiltonian, the boson Hubbard model, I discuss the path integral representation of it followed by scaling theory, including arguments that the conductivity is universal at the critical point. The last part of the lectures describes Monte Carlo simulations, which give predictions for universal quantities at the transition.

## 1. INTRODUCTION

Most studies of phase transitions have concentrated on systems where the transition is driven by a change in temperature, *i.e.* starting from an ordered state at low temperature, *thermal fluctuations* become more and more important as the temperature is raised until, at the transition temperature $T_c$, the order is destroyed. There are, however, interesting systems in which the transition is driven by *quantum* fluctuations. These are transitions which occur at $T = 0$ (so there are no thermal fluctuations) when some parameter in the system is changed. A transition occurring at a finite temperature always has the behavior of a *classical* thermally driven transition sufficiently close to $T_c$, though, if $T_c$ is small, there may be a range of temperature where quantum behavior is seen. A typical phase diagram is sketched in Fig. 1. Here $\Delta$ is some parameter in the system (in these lectures it will be disorder) which drives $T_c$ to zero at a critical value, $\Delta_c$. The range over which classical critical behavior is observed gets smaller and smaller as $\Delta \to \Delta_c$. To understand this, note that fluctuations are classical if $\hbar\omega_c < k_B T$, where $\omega_c$ is a characteristic frequency of the fluctuation. As a critical point is approached, $\omega_c$ goes to zero (critical slowing down), as discussed in Sec. 7 below. Hence, sufficiently close to the critical point, fluctuations are always classical if $T_c > 0$ , though the size of the critical region will go to zero as $T_c \to 0$. The quantum transition that we are studying here corresponds to fixing $T = 0$ and varying $\Delta$ through $\Delta_c$. Note that in the vicinity of $\Delta_c$, quantum critical behavior will also be observed at low but finite $T$ provided the system is not too close to the finite-$T$ transition.

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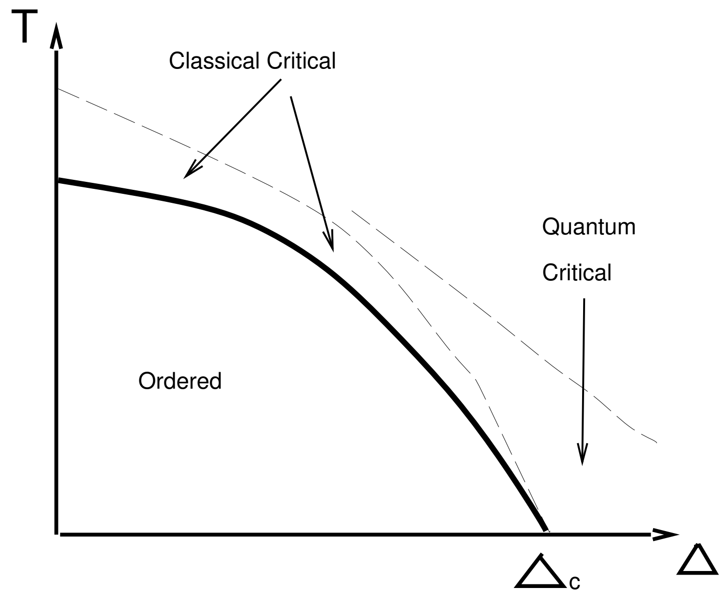


Figure 1. A sketch of a typical phase diagram showing a transition at finite-$T$ being driven to $T = 0$ by varying some parameter $\Delta$, (disorder in the examples studied here). The regions of classical and quantum critical behavior are shown. There are also regions of classical and quantum critical behavior in the ordered state, but these are not shown.

Quantum phase transitions in clean systems do not differ all that much from classical phase transitions, and generally correspond to the behavior of some classical system in a higher dimension, see *e.g.* Ref. [1]. However, the physics of quantum phase transitions in *disordered systems* is quite different from that of classical systems, and is relatively poorly understood. For example, in conventional transitions, much information has been obtained by the expansion of critical exponents in powers of $\epsilon$, where $\epsilon = d_u - d$ and $d_u$ is the upper critical dimension above which mean field exponents are obtained. However, for quantum transitions in disordered systems, which will be the focus of these lectures, the $\epsilon$-expansion does not give sensible results: in fact it is not clear what is the value of $d_u$ or even if this concept makes sense at all [1]. As a result, a lot of emphasis will be placed on numerical methods such as Monte Carlo simulations.

## 2. LOCALIZATION THEORY

The quantum transition which has received the most attention has been metal-insulator transition in Fermi systems [2]. There, one considers electrons (non-interacting in most theories) moving in a random potential. If the disorder in the potential is weak, then, for dimension, $d$, greater than or equal to 3, some of the the wave functions are *extended* over the whole sample, *i.e.* they have amplitude of order $1/N^{1/2}$ (where $N$ is the number of sites) *everywhere*. Other states, (in practice those in the tails of the band), are localized, *i.e.* they have amplitude of order unity on a *finite* number of sites centered on a particular point in the sample. There are a energies, called the mobility edge, $\epsilon_c$, which separates localized from extended states. For each energy, there is a localization length $\xi_1(\epsilon)$ (finite in the localized regime) which is the spatial extent of a typical state at that energy. One has $\xi_1(\epsilon) \to \infty$ as $\epsilon \to \epsilon_c$ from the insulating side.

If the fermi energy, $\epsilon_F$, is moved through the mobility edge, at $T = 0$, then the system will undergo a transition from an insulator (when $\epsilon_F$ is in the region of localized states) to a metal (when $\epsilon_F$ is in the region of extended states). Although this general picture is well established [2], the details are not completely clear. For example, there is no general agreement as to the value of the critical exponent with which the conductivity vanishes as one approaches the transition from the metallic side [2].

For $d = 2$, the situation is rather different. It turns out [2] that an arbitrarily weak random potential localizes all the states. The localization length can be huge if the disorder is small; for example at the band center, the localization length diverges exponentially as the strength of the disorder goes to zero. Consequently the conductivity is always zero at $T = 0$ in a disordered $2d$ system because the electrons are trapped in localized states. However, at finite $T$, the conductivity is finite because of inelastic collisions which can scatter an electron from one localized state to another, so the electron can eventually move from one end of the sample to another, *i.e.* conduct. To summarize, there is *no* metallic state in $2d$ at $T = 0$ for systems with disorder. This will be important in what follows.

The theory of localization has been worked out in some detail for the case of non-interacting fermions, where a mapping can be made to a field theory which is amenable to systematic treatment. The situation is much more complicated when interactions are included [2], since it has not been possible to cast the problem as a field theory. It appears, though, that repulsive interactions enhance the tendency towards localization. Thus there will still be no metallic state in $2d$. Attractive interactions lead to superconductivity which will be discussed at length in these lectures. We shall then argue that a description in terms of *bosons* is more appropriate [1, 3]. Note that superconductivity can exist at finite $T$ in $2d$. Even though there is not true long range order, the "stiffness" with respect to twisting the phase (*i.e.* the superfluid density) is non-zero [4]. The finite temperature superfluid to normal transition (not to be confused with the quantum transition which is the focus of these lectures) is described by the Kosterlitz-Thouless [4] theory.

Until recently far less attention had been paid to bosons in a random potential than to fermions. This is despite there being experimental realizations. One example is $^4$He absorbed in porous media or on various substrates [5]. Another example, which we shall discuss in detail in these lectures, is the superconducting transition, for which the Cooper pairs may act, at least approximately, as bosons [1, 3]. Just as for fermions, one expects that if the fluctuations in the random potential are large, the bosons will be localized, but that if the random potential is weak, then the bosons will be mobile and so can conduct. One difference from fermions is that one has to include repulsive interactions from the start otherwise all the particles will condense in the lowest localized eigenstate. Thus repulsion spreads the bosons away from the points of minimum potential which turns out to have something of a *de*-localizing tendency. Consequently, it appears that one *can* have a conducting state of (repulsive) bosons in a disordered $2d$ system, even though one cannot have a conducting state of (repulsive) fermions, as discussed above. This will be of crucial importance in what follows.

For pure Bose systems, the conducting phase at zero temperature is apparently always superfluid so the conducting-insulator transition corresponds to the onset of superfluidity [1] (or to superconductivity if the bosons are charged). Thus, in contrast to the case of Fermi systems, there is a natural order parameter for the Bose problem, associated with off-diagonal long-range order of the phase of the superfluid.

Let us now discuss what might happen in a Fermi system with (net) attractive interaction if one increases the degree of disorder at $T = 0$. With no disorder, the system is a superconductor. On the other hand, in the limit of very strong disorder, the system will be in a localized, insulating state. For $d \geq 3$ there is also the possibility of a metallic state, which might intervene at intermediate values of the disorder. Hence, the change from superconductor to insulator at $T = 0$ may proceed in two stages for $d \geq 3$: (i) a transition from superconductor to metal, followed by (ii) a metal-insulator transition. Presumably the latter would be of the conventional type that has been much studied [2]. However, the quantum superconductor-metal transition, has not yet received much attention. By contrast the situation in $d = 2$ is quite different because there can be no metallic phase. Thus, the transformation from superconductor to insulator is expected to occur in a single transition. This transition will be the focus of the rest of these lectures.

## 3. SUPERCONDUCTOR-INSULATOR TRANSITION IN 2D

To which universality class does the superconductor-insulator transition in $2d$ belong? It is hypothesized [3] that it can be modeled by interacting bosons of charge $e^\star = 2e$ moving in a random potential. Loosely speaking, we assume that the Cooper pairs have already formed prior to the transition to superconductivity. This is reasonable, but one may need to be quite close to the transition for it to occur: the formation of Cooper pairs indicates that short range order is starting to develop. Short range order at the classical finite-$T$ bulk ($3d$) transition in conventional low-$T_c$ superconductors only develops very close to $T_c$. However, fluctuations extend over a wider range of temperatures in $2d$ systems, which is the case considered here. Furthermore fluctuation effects are much more important in high-$T_c$ materials, so critical behavior at the superconductor-insulator transition in $2d$ will probably be easier to see in these materials. More specifically, we expect that the Cooper pair size (defined below) will remain finite at the transition, even though the pair coherence length, $\xi$, diverges. Thus, on the scale of $\xi$, the Cooper pairs look like point bosons. Of course, the size of the Cooper pair is large compared with the separation between the pairs, so *microscopically* one cannot represent the pairs as bosons, because the wave function must be antisymmetrized between electrons in different but overlapping pairs. However, one argues that the details of the physics on small length scales (such as the pair size or pair separation) are not relevant for critical phenomena. Rather what matters is the *divergent* length scale or scales. We shall argue that the only divergent length scale in this problem is the boson coherence length, $\xi$. To be more concrete consider [3] the spatial dependence of the equal time pair correlation function

$$C_2(\mathbf{x}_1, \mathbf{x}_1', \mathbf{x}_2, \mathbf{x}_2') = \langle c_\uparrow^\dagger(\mathbf{x}_1) c_\downarrow^\dagger(\mathbf{x}_1') c_\downarrow(\mathbf{x}_2) c_\uparrow(\mathbf{x}_2') \rangle \quad , \tag{1}$$

where $c_\sigma(\mathbf{x})$ is an electron annihilation operator with spin $\sigma$. In the superconducting phase as $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$ (with $|\mathbf{x}_1 - \mathbf{x}_1'|$ and $|\mathbf{x}_2 - \mathbf{x}_2'|$ finite) $C_2$ in Eq. (1) factorizes as

$C_2 \simeq \psi^\star(\mathbf{x}_1 - \mathbf{x}_1')\psi(\mathbf{x}_2 - \mathbf{x}_2')$ where $\psi(\mathbf{x})$ is the Cooper pair wave function. One expects that $\psi(\mathbf{x}) \sim \exp(-|\mathbf{x}|/\xi_p)$, which defines the Cooper pair size $\xi_p$. In the insulating side, but close to the transition, as $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$, $C_2$ should still factorize, taking the form

$$C_2 \simeq e^{-|\mathbf{x}_1 - \mathbf{x}_2|/\xi} \psi^\star(\mathbf{x}_1 - \mathbf{x}_1')\psi(\mathbf{x}_2 - \mathbf{x}_2') \quad . \tag{2}$$

The Cooper pair size can then be defined from $\psi(\mathbf{x})$ even in the insulating phase. We hypothesize that upon approaching the $(T = 0)$ transition to superconductivity, $\xi$ diverges but $\xi_p$ stays finite.

From Eq. (2) it follows that in the insulating phase $\xi$ is essentially a Cooper *pair* (or boson) localization length. The single electron localization length for electrons at the fermi energy can also be defined in terms of the (frequency dependent) single electron (rather than pair) correlation function $C(\mathbf{x}, \mathbf{x}', E - \mu)$:

$$C(\mathbf{x}, \mathbf{x}', 0) \simeq e^{-|\mathbf{x}|/\xi_1} \quad . \tag{3}$$

This single electron localization length, while clearly finite in the insulating phase is *also* expected to be finite in the in the superconducting phase, since electrons can only propagate coherently as pairs. In a Fermi liquid phase, $\xi_1$ is infinite but since the transition does not involve a Fermi liquid, we expect that $\xi_1$ will stay finite here even at the transition. Thus it seems at least plausible that $\xi$ is the only divergent length and so Bose physics should describe the critical behavior.

The hypothesis that the superconductor-insulator transition is in the same universality class as bosons moving in a random potential has been shown to be true for the $1d$ case by Giamarchi and Schulz [6].

In the rest of these lectures, we will study the superconductor-insulator transition of $2d$ bosons in a random potential. We will aim to determine universal quantities, such as critical exponents. Much of the interest in this problem has stemmed from the claim [7] that the conductivity, at the critical point, $\sigma^*$, is predicted to be finite (and universal). The universal value is a numerical constant times the quantum of conductance, $G_Q$, where

$$G_Q^{-1} \equiv R_Q = \frac{h}{(2e)^2} \simeq 6.45 k\Omega \quad . \tag{4}$$

It is important to note that conductance and conductivity have the same dimensions in $2d$ and that the conductance per square, $\sigma_\square$, is equal to the conductivity. Note also that on the insulating side $\sigma = 0$ and on the superconducting side $\sigma = \infty$ so it is only at the critical point that one can have metallic behavior.

Experiments have been carried out which seems to confirm this picture [8–10]. Films which are relatively clean show a resistance which drops as $T$ is reduced, indicating superconductivity. Dirty samples show a resistance which increases as $T$ decreases, indicating insulating behavior, while samples on the bordeline between these two regions have a resistance per square close to $R_Q$ as expected. Different experiments seem to get somewhat different values for the critical resistivity, $R_\square^*$, perhaps because they are not all in the critical regime. It is also possible to tune through the superconductor-insulator transition by varying a magnetic field [11, 12].

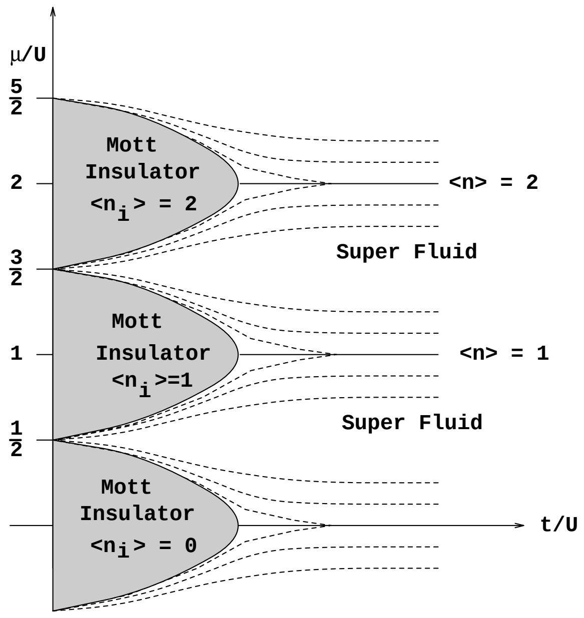


Figure 2. The phase diagram of the boson Hubbard model in the absence of disorder. There are two phases, the Mott insulator and the superfluid phase. The Mott insulator has zero compressibility. A locus of constant integer density is shown. It intersects the phase boundary at the tip of the lobe of the Mott insulator. Also shown is the locus of points for constant non-integral density, $n$. This always lies in the superfluid region and hits the $\mu$ axis where $\mu/U$ is an the smallest integer less than $n$.

## 4. MODEL HAMILTONIAN

If we accept that universal properties at the superconductor-insulator transition are given by bosons moving in a random potential, a suitable Hamiltonian is the boson Hubbard model [1]

$$\mathcal{H} = -\sum_i(-zt + \mu + \delta\mu_i)\hat{n}_i + \frac{U}{2}\sum_i \hat{n}_i\hat{n}_i - t\sum_{\langle i,j\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) \quad , \tag{5}$$

where $\hat{n}_i = b_i^\dagger b_i$ and $b_i$ is a boson field operator on site $i$ which satisfies the standard commutation relations $[b_i, b_j^\dagger] = \delta_{ij}$. Here $t$ is the strength of the hopping between nearest-neighbor sites sites $i$ and $j$, and $z$ is the coordination number. The average chemical potential $\mu$ fixes the boson density, whereas $\delta\mu_i$ is a random on-site potential which varies between $\Delta/2$ and $-\Delta/2$. The on site soft-core repulsion has strength $U$.

## 5. PHASE DIAGRAM

Following Ref. [1] we now discuss the expected phase diagram for this model. It is easiest to consider first the case of no disorder. In the trivial limit of $t = 0$ each site is occupied by the integer number $n$ of bosons which minimizes the on-site energy

$$\epsilon(n) = -\mu n + \frac{U}{2} n^2 \quad . \tag{6}$$

Thus, for all value of $\mu$ in the interval $n - 1/2 < \mu/U < n + 1/2$, there are exactly $n$ bosons on each site. The number of bosons cannot be negative so $n = 0$ for $\mu < 0$. Now consider $\mu/U$ between $n - 1/2$ and $n + 1/2$ and turn on a weak hopping $t$. Then, one argues [1] that if $t$ is sufficiently small, the kinetic energy gained by allowing the particle to hop is insufficient to compensate for the potential energy loss in adding a particle to one site and subtracting a particle from another site. Thus, one expects that there is a finite region in the $\mu - t$ plane, see Fig. 2, in which the number of particles is fixed at exactly $n$ per site. These regions represent insulating states, characterized by the existence of an energy gap $E_g$ for creation of particle or hole excitations.

Note that these states are different from the insulating states that we discussed earlier. There, localization was due to disorder. Here, there is no disorder, but the state is, nonetheless, insulating because of correlation effects. The argument for these states to be insulating is the same as that used by Mott to understand why transition metal oxides (which have an unfilled $d$-band) are insulating. We shall therefore call these regions Mott insulators. Note that the superfluid region extends all the way down to $t = 0$ at half-integer values of $\mu/U$ since, for $t = 0$, the energy to occupy a site with $n$ particles is the same as that of occupying it with $n - 1$ particles.

If $\langle \hat{n} \rangle$ is an integer, it is not difficult to see [1] that the locus of such points is a line which goes through the tips of the "Mott lobes" in the phase diagram. For non-integer densities, between $n$ and $n+1$, the the locus of points of constant density is always in the superfluid region because the bosons can hop around with no energy cost, and terminates on the $\mu$ axis in betwee two of the Mott lobes.

If one does not keep the density fixed, it is possible to cross the transition line from Mott insulator to superfluid at non-integer values of the density, *i.e.* away from the lobes in Fig. 2. We shall see later that the critical behavior in this case is different from that obtained if the density is integral.

Adding disorder makes the model much more complicated. The main new feature [1] is another insulating phase, which occurs because the bosons can be localized due to disorder. This is called the "Bose glass" phase. How can one distinguish the Bose glass and Mott insulating phases? After all both have zero conductivity. The difference lies in the energy gap for adding or removing particles. We have already mentioned that this is finite in the Mott insulator. Another way of saying this is that the compressibility,

$$\kappa \equiv \frac{1}{n^2} \frac{\partial n}{\partial \mu} \tag{7}$$

is zero, because changing the chemical potential (such that it still lies in the gap) does not change the density. By contrast, the Bose glass does not have a gap because we assume a continuous distribution of the $\delta\mu_i$ and so one expects there to be some states arbitrarily close to the Fermi energy. Consequently the compressibility is finite. Thus there is a qualitative difference between the Bose glass and Mott insulator; they are separate phases and a sharp transition can occur between them. A possible topology for the phase diagram is shown in Fig. 3 for weak disorder.

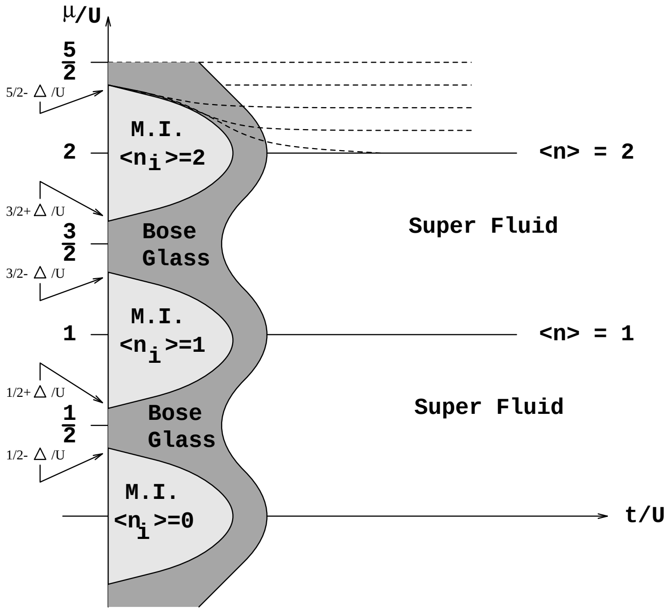


Figure 3. Possible $T = 0$ phase diagram of the boson Hubbard model with weak disorder. In addition to the Mott insulator and superfluid phases shown in Fig. 2, there is also a Bose glass phase. This differs from the Mott insulator in having a finite compressibility. For strong disorder, the Mott insulator phase disappears.

The experimentally observed superconductor-insulator transition is expected to be in the same universality class as the Bose glass - superfluid transition in Fig. 3 (for particles with long-range Coulomb interactions).

## 6. PATH INTEGRAL FORMULATION

In order to understand the symmetries of the system, and hence possible universality classes, it is useful to set up the problem as a path integral. This is the approach one takes to set up the $\epsilon$-expansion, discussed below. We consider first the case of no disorder. In the absence of hopping, the Hamiltonian (5) becomes a set of on-site terms can easily be diagonalized. Progress can therefore be made by factoring by factoring out the hopping term, *i.e.* writing [1]

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{8}$$

where

$$\mathcal{H}_1 = -t \sum_{\langle i,j \rangle} (b_i^\dagger b_j + b_j^\dagger b_i) \tag{9}$$

is the hopping term and $\mathcal{H}_0$ is the on-site term, the rest of Eq. (5). One can express the partition function $Z = \mathrm{Tr}\exp(-\beta\mathcal{H})$ in the interaction representation as

$$Z = Z_0 \left\langle T_\tau \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \mathcal{H}_1(\tau)\right]\right\rangle_0 \quad , \tag{10}$$

where $Z_0 = \mathrm{Tr}\exp(-\beta\mathcal{H}_0)$,

$$\mathcal{H}_1(\tau) = e^{\tau\mathcal{H}_0}\mathcal{H}_1 e^{-\tau\mathcal{H}_0} \quad , \tag{11}$$

and the average is taken over the ensemble defined by $\mathcal{H}_0$. Here $T_\tau$ is the imaginary time ordering operator. Because of the analogy between the Boltzmann factor in statistical mechanics, $\exp(\beta\mathcal{H})$, and the time evolution operator in quantum mechanics, $\exp(it\mathcal{H}/\hbar)$, $\tau$ can be thought of as an imaginary time, and time or frequency dependent Greens functions can be obtained by analytic continuation of the imaginary time correlation functions given by this action. A complex $c$-number field $\psi_i(\tau)$ can now be introduced [1] to decouple the hopping term with the result that

$$Z = Z_0 \int \prod_i D\psi_i(\tau) D\psi_i^\star(\tau) \exp\left[-S[\psi]\right] \quad , \tag{12}$$

where the action is

$$S[\psi] = \sum_{i,j} \int d\tau (t^{-1})_{ij} \psi_i^\star(\tau)\psi_j(\tau) - \sum_j \ln \left\langle T_\tau \exp\left[\int d\tau \psi_i(\tau) b_i^\dagger(\tau) + \mathrm{H.c.}\right]\right\rangle_0 \quad , \tag{13}$$

where $(t^{-1})_{ij}$ is the inverse of the hopping matrix. One can show that $\langle b^\dagger\rangle \propto \langle\psi\rangle$, so $\psi$ is the order parameter for Bose-Einstein condensation. The average in Eq. (13) can now be evaluated as an expansion in powers of the $\psi_i(\tau)$, with the result [1]

$$S[\psi] = \frac{1}{2}\int_{k,\Omega} (k^2 + \Omega^2 + r)|\psi(k,\Omega)|^2 + g\int_{k,\Omega} i\Omega|\psi(k,\Omega)|^2 + u\int_{x,\tau} |\psi|^4 + \cdots \quad . \tag{14}$$

Here $\Omega$ is the imaginary frequency corresponding to the imaginary time $\tau$, and $x$ and $\tau$ have been rescaled to fix the coefficients of $k^2\psi^2$ and $\Omega^2\psi^2$ to be $\frac{1}{2}$. At finite $T$, $\Omega$, the imaginary frequency, runs over the discrete Matsubara frequencies, $\Omega = \omega_n \equiv 2\pi n k_B T$, where $n$ is an integer, but at $T = 0$, the case of interest here, $\Omega$ is a continuous variable. Higher order terms in Eq. (14) are expected to be irrelevant at the critical point.

It is natural that the imaginary term $gi\Omega\psi^2$ appears, since this is first order in (imaginary) time, and a first order order time derivative occurs in the boson Schrödinger equation. What is perhaps a bit more surprising, is that the coefficient, $g$, vanishes at the tips of the Mott lobes [1] in Fig. 2, where the particle density is an integer, because of a particle-hole symmetry. Noting that $\Omega$ then appears on the same footing as $k$ in Eq. (14), one sees that $S$ is the action of a *classical* 2-component system (XY model) in $d+1$ dimensions, the extra dimension corresponding to imaginary time. This, then, is the universality class of the Mott insulator - superfluid transition at integer density. Away from integer density, $g$ is non-zero and the universality class for this "generic" transition is different, see Ref. [1] for details.

The situation is a bit more more complicated when we include disorder. One must average the free energy, not the partition function or Hamiltonian, over the disorder. This is accomplished using the replica trick, [13] where one uses the identity

$$[\ln Z]_{av} = \lim_{n\to 0} \frac{[Z^n]_{av} - 1}{n} \quad , \tag{15}$$

where $[\cdots]_{av}$ denotes an average over the disorder. The $n$-th power of the partition function is evaluated by considering $n$ identical replicas of the system. Averaging over the disorder can now be accomplished because the random variables appear in the exponential of $Z^n$ (rather than being buried inside the log). This leads to additional terms in the action, involving cumulants of the disorder distribution, which couple different replicas. Keeping only the lowest order (*i.e.* second) cumulant of the distribution of the $\delta\mu_i$ one obtains [1]

$$S = \sum_{\alpha=1}^{n} \left[ \int_{k,\Omega} \frac{1}{2}(k^2+\Omega^2+r)|\psi_\alpha(k,\Omega)|^2 + g\int_{k,\Omega} i\Omega|\psi_\alpha(k,\Omega)|^2 + u\int_{x,\tau}|\psi_\alpha|^4 \right]$$

$$-w\sum_{\alpha,\beta=1}^{n}\int_{x,\tau,\tau'}|\psi_\alpha(x,\tau)|^2|\psi_\beta(x,\tau')|^2 \quad . \tag{16}$$

Do not confuse the number of replicas with the density. The disorder enters in the term involving $w$, which is proportional to the variance of $\delta\mu_i$. Note that it couples the replicas and is *non-local* in time because the disorder is *quenched*, *i.e.* time independent.

One can now try to compute the properties of the action (6) by $\epsilon$-expansion methods. The coefficient, $g$, of the $i\Omega|\psi^2|$ term is non-zero in general and so one can presumably then neglect the $\Omega^2|\psi|^2$ term. One has to imagine that the $w$ term is non-local in $\epsilon_\tau$ dimensions, and local in the remaining $d$. One can then generate a double expansion [14] in powers of $\epsilon_\tau$ and $\epsilon = 4 - d - \epsilon_\tau$. The physically interesting case here is $\epsilon_\tau = \epsilon = 1$. Unfortunately one finds that the effective value of $w$ on long scales, obtained by integrating out short wavelength degrees of freedom, grows and the perturbative renormalization group approach breaks down [1]. As we shall discuss in Sec. 7 below, this may be because there is *no* upper critical dimension above which mean field exponents are obtained. Because the $\epsilon$-expansion does not work, numerical approaches, such as Monte Carlo simulations, have to be used.

## 7. SCALING RELATIONS

Scaling theories have been very useful to describe finite-temperature transitions. Here we discuss how these ideas can be applied to quantum transition, especially for the model of bosons discussed above. Our discussion follows the arguments of Ref. [1]. We need to introduce a parameter, $\delta$, which is a measure of the distance from the transition. For example, for the integer density Mott to superfluid transition with no disorder (see Fig. 2),

$$\delta \sim t - t_c \tag{17}$$

where $t_c$ is the value of the hopping parameter, $t$, at the transition. The central assumption in the scaling theory is that there is a single important diverging length scale, and a single characteristic frequency, $\omega_c$, which vary as

$$\xi \sim \delta^{-\nu} \tag{18}$$

$$\omega_c \sim \xi^{-z} \sim \delta^{z\nu} \quad , \tag{19}$$

which defines the correlation length exponent, $\nu$ and the dynamical exponent $z$. As mentioned above, the action can be viewed as a classical action in $(d+1)$ dimensions.

The lattice size will be $L$ in the space dimensions (with $N = L^d$) and $\beta\hbar$ in the imaginary time direction, so the $(d+1)$-dimensional "volume" is

$$V = L^d \beta\hbar \quad . \tag{20}$$

We are interested in the zero-temperature thermodynamic limit, $L \to \infty, \beta \to \infty$. Note that the time scale $\omega_c^{-1}$ can be thought of as a correlation "length", $\xi_\tau$, in the time dimension,

$$\xi_\tau \equiv \omega_c^{-1} \sim \xi^z \quad . \tag{21}$$

Note also that the $T = 0$ *energy* per site,

$$f \equiv - \lim_{\beta\to\infty} \lim_{L\to\infty} \frac{\hbar}{(\beta\hbar)L^d} \ln Z = -\frac{\hbar}{V} \ln Z = -\frac{\hbar}{V} \ln \mathrm{Tr} e^{-S} \quad , \tag{22}$$

corresponds to the *free energy* density of the classical $(d+1)$-dimensional system, whose effective Hamiltonian is the action $S$, and with the role of temperature replaced by $\hbar$. With this analogy, the scaling theory can then formulated. If hyperscaling holds, the singular part of the free energy per correlation volume, $\xi^d \xi_\tau \sim \xi^{d+z}$, is of order $\hbar$, so

$$f_s \sim \xi^{-(d+z)} \sim \delta^{(d+z)\nu} \quad . \tag{23}$$

Correlation functions should also exhibit scaling near the transition. For example, the order-parameter susceptibility defined by

$$\chi(\mathbf{r}, \tau) \equiv \left[ \langle T_\tau b_r(\tau) b_0^\dagger(0) \rangle \right]_{av} \tag{24}$$

should only be a function of the combinations $r/\xi$ and $\tau/\xi^z$, apart from an overall power law, *i.e.*

$$\chi(\mathbf{r}, \tau) \sim \frac{1}{r^{d+z-2+\eta}} \tilde{\chi}(r/\xi, \tau/\xi^z) \quad , \tag{25}$$

which defines the exponent $\eta$. At criticality ($\xi = \infty$) one has a power law variation

$$\chi(\mathbf{r}, 0) \sim \frac{1}{r^{d+z-2+\eta}} \quad , \tag{26}$$

$$\chi(0, \tau) \sim \frac{1}{\tau^{(d+z-2+\eta)/z}} \quad , \tag{27}$$

which is obtained from Eq. (25) by requiring that the dependence on $\xi$ must cancel.

In the superfluid phase, it costs energy to impose a twist on the phase of the condensate, $\theta$. The superfluid state therefore has a rigidity associated with it. More precisely, the extra energy density (which corresponds to the free energy density of the equivalent $(d+1)$-dimensional model) is proportional to $(\nabla\theta)^2$ at long wavelengths, *i.e.*

$$\delta f_s/\hbar = \frac{1}{2} \frac{\hbar}{m^2} \rho_s (\nabla\theta)^2 \tag{28}$$

where $\rho_s$ is called the superfluid density and $m$ is the boson mass. The reason for this name is that one can show that the mass current density is $\rho_s(\hbar/m)\nabla\theta$. Just as in the derivation of the hypercaling relation for the free energy, Eq. (23) above, we assume that $\delta f_s$ integrated over a correlation volume, $\xi^{d+z}$, is a constant (of order $\hbar$) and we note that, in equilibrium, the phase changes significantly on a scale of $\xi$, so, from Eq. (28) we get

$$\rho_s \sim \delta^{\zeta} \quad , \tag{29}$$

with

$$\zeta = (d + z - 2)\nu \quad . \tag{30}$$

This is a generalization of the familiar Josephson scaling relation $\rho_s \sim \delta^{(d-2)\nu}$, for the classical transition but with $d$ replaced by $d + z$. In fact, all hyperscaling relations (those which involve the space dimensionality $d$) for classical systems go over to the quantum case with $d$ replaced by $d + z$.

One can also derive a similar relation for the compressibility [1] by applying a twist along the imaginary time direction instead of the space direction. As shown by Josephson a change of phase of the order parameter with respect to real time gives a change in the chemical potential, $\delta\mu = \hbar\partial\theta/\partial t$. Noting that $\kappa = -\partial^2 f/\partial\mu^2$ one has

$$\delta f_s/\hbar = \frac{1}{2}\hbar\kappa(\partial_\tau\theta)^2 \quad . \tag{31}$$

In equilibrium, significant fluctuations in $\theta$ occur over an imaginary time $\xi_\tau \sim \xi^z$ so, assuming as above that $\delta f_s/\hbar$ integrated over a correlation volume is of order unity, we get

$$\kappa \sim \xi^{-(d+z-2z)} \sim \xi^{-(d-z)} \quad . \tag{32}$$

In Ref. [1] it is argued that the compressibility stays finite at the transition and

$$z = d \quad . \tag{33}$$

A rigorous bound on $\nu$ is [15, 16]

$$\nu \geq 2/d \quad , \tag{34}$$

and Ref. [1] gives arguments that

$$\eta \leq 2 - d \quad . \tag{35}$$

At conventional classical transitions, the critical exponents approach their mean-field values *continuously* as the dimension approaches the upper critical dimension, $d_u$, from below. This is the basis of the $\epsilon$-expansion approach to calculating critical exponents. However, a continuous variation of exponents at $d_u$ is not possible here for the following reason. The mean field values are $\zeta = 1, \nu = 1/2$ which, if the exponents are continuous, imples from Eq. (30) that $d_u = 4 - z < d$, since $z > 0$. But the inequality in Eq. (34) with $\nu = 1/2$ shows that $d_u > 4$ so there is an inconsistency. Consequently, it does not seem possible to form an $\epsilon$-expansion for quantum transitions with disorder, and, as a result, our estimate for critical exponents and other univeral quantities will come from Monte Carlo simulations. It is possible [1] that there is *no* upper critical dimension and $z = d$ for all $d$. Another possibility is that the exponents jump *discontinuously* to mean field values at the upper critical dimension (which must be greater than 4 and might be non-universal, depending, for example, on the strength of the disorder).

## 8. UNIVERSAL CONDUCTIVITY

In critical phenomena, it is well known that exponents are universal. It is also moderately well known that that there are universal amplitude ratios. For example if the singular part of the specific heat, $C_{sing}$, varies as $C_{sing} = A_{\pm}|t|^{-\alpha}$, at a classical finite-$T$ transition, where $t$ is the reduced temperature and "+" and "−" refer to $T > T_c$ and $T < T_c$ respectively, then the ratio of critical amplitudes $A_+/A_-$ is obviously dimensionless and also turns out to be universal [17, 18].. What is less well known is that *all* dimensionless combinations of critical amplitudes are universal [17, 18], which is known as two scale factor universality. For example, at a quantum transition, consider

$$\frac{f_s}{\hbar}\xi^d\xi_\tau = C \tag{36}$$

where, from Eq. (23), $C$ is a constant. Furthermore, $C$ is dimensionless, and is therefore universal from two scale factor universality. Now consider the conductivity, $\sigma$. From electrodynamics in the gauge with zero scalar potential, we have $J = \sigma E$, $J = \partial F/\partial A$, $E = -\partial A/\partial t$, so, schematically

$$\sigma(\omega) = \frac{1}{-i\omega}\frac{\partial^2 F}{\partial A_x^2} \quad , \tag{37}$$

where $\omega$ is the real frequency. We can reepress $\sigma$ in terms of the imaginary frequency $\Omega = -i\omega$, which appears in the path integral formulation, as

$$\sigma(\omega = i\Omega) = \frac{1}{\Omega}\frac{\partial^2 f}{\partial A_x^2} \quad . \tag{38}$$

The vector potential appears in the usual way, *i.e.* in Eq. (28) we make the replacement

$$\nabla\theta \to \nabla\theta - \frac{e^\star}{\hbar}\mathbf{A} \tag{39}$$

where $e^\star \equiv 2e$ is the charge of the bosons, since $\theta$ is the phase of the condensate, which is like a quantum mechanical wave function, and this form is the minimal coupling which ensures gauge invariance. From Eqs. (28), and (37)-(39) one gets [3]

$$\sigma(i\Omega) = \frac{e^{\star 2}}{\hbar}\frac{1}{\Omega}\rho(\Omega) \tag{40}$$

where

$$\rho(\Omega) = \frac{1}{V}\frac{\partial^2 \log Z}{\partial(\nabla\Theta_\Omega)^2} \tag{41}$$

is a generalized stiffness giving the change in free energy on applying and external twist, $\nabla\Theta_\Omega$, along one of the space directions but with a periodicity $\Omega$ in the imaginary time direction. $\rho(\Omega)$ is related to the superfluid density in Eq. (28) by

$$\rho(\Omega = 0) = \frac{\hbar}{m^2}\rho_s \tag{42}$$

and can be expresssed as a Kubo formula [3].

The Josephson scaling relation, Eq. (29) generalizes to finite $\Omega$ as

$$\rho(\Omega) = \xi^{-(d-2)}\xi_\tau^{-1}\tilde{\rho}(\Omega\xi_\tau) \tag{43}$$

where we have made the usual scaling assumption that dependence on $\Omega$ only enters in the combination $\Omega\xi_\tau$, where $\xi_\tau$ is the correlation "length" in the imaginary time direction. By this hypothesis $\tilde{\rho}$ is a universal scaling function. At the critical point, the dependence on $\xi$ and $\xi_\tau$ must drop out. Considering now just the case, $d = 2$, we must have $\tilde{\rho}(x) \sim x$ for large $x$, and hence

$$\rho(\Omega) = \Omega Q \qquad (d = 2) \quad , \tag{44}$$

where $Q = \lim_{x\to\infty}, \tilde{\rho}(x)/x$ is universal. Substituting into Eq. (40) we get, at the quantum critical point,

$$\sigma(\omega \to 0) = C G_Q \tag{45}$$

where $G_Q$, given by Eq. (4), is the quantum of conductance, $e^\star$, the boson charge is given by $e^\star = 2e$, and $C = 2\pi Q$ is a *universal* [7] constant. Note that in Eq. (45) the exponent is zero (so the conductivity is finite at the critical point) and the amplitude $C$ is dimensionless hence is universal. Therefore the conductivity at the superconductor-insulator transition in two-dimensions is universal (in units of $\sigma_Q$). Note that the conductivity being dimensionless and universal is special to $d = 2$. From Ohms law one sees that in $d$ dimensions, the conductivity has dimensions $L^{2-d} \times \sigma_Q$ and so extra factors of length appear for $d \neq 2$. A related example of a universal amplitude, is the jump [19] in $\rho_s/T_c$ at the finite-$T$ Kosterlitz-Thouless transition in the $2d$ XY model [4] (this model describes the *finite-T* transition of superconducting films).

## 9. TRANSFORMATION OF THE HAMILTONIAN

The Monte Carlo simulation uses a discrete-time version of the path integral approach discussed in Sec. 6. One follows the trajectories, or "world lines", of the particles in imaginary time. In order to make the model convenient for Monte Carlo simulations we use a version of the path integral method which effectively only includes phase fluctuations of the bosons [20, 21]. One finds

$$Z = \sum_{\mathbf{J}}{}' \exp\left\{-\frac{1}{K}\sum_{(\mathbf{r},\tau)}\left[\frac{1}{2}\mathbf{J}^2_{(\mathbf{r},\tau)} - (\tilde{\mu} + \tilde{v}_{\mathbf{r}})J^\tau_{(\mathbf{r},\tau)}\right]\right\} , \tag{46}$$

where the $J^\nu, \nu = x, y$, or $\tau$, are integer variables which take values from $-\infty$ to $\infty$, on the links of the (2+1) dimensional lattice, specified by a space coordinate $\mathbf{r}$ and a time coordinate $\tau$, and the prime indicates that the $J^\nu$ satisfy the zero-divergence condition $\sum_\nu J^\nu_{(\mathbf{r},\tau)} = 0$, where $\nu$ runs over $\pm x, \pm y, \pm\tau$. Here $\tilde{\mu}$ and $\tilde{v}$ are given by

$$\begin{aligned} \tilde{\mu} &= \mu/U \\ \tilde{v}_{\mathbf{r}} &= \delta\mu_{\mathbf{r}}/U \ , \end{aligned} \tag{47}$$

and $K$ is a dimensionless coupling constant which has to be adjusted to bring the system to the critical point. Varying $K$ corresponds to changing the ratio $t/U$ in the boson Hubbard model, keeping $\mu/U$ and $\Delta/U$ fixed. Noting the invariance of the action under

$$\begin{aligned} J^\tau &\longrightarrow J^\tau + 1 \\ \tilde{\mu} + \tilde{v} &\longrightarrow \tilde{\mu} + \tilde{v} + 1, \end{aligned} \tag{48}$$

we take for simplicity the "largest possible" disorder by choosing $\tilde{\mu} = 1/2, \tilde{\Delta} \equiv \Delta/U = 1/2$. The average particle density is then $1/2$.

The partition function, Eq. (46) can be written in terms of an effective (2+1)D classical Hamiltonian or action, given by

$$H_V = \frac{1}{K} \sum_{(\mathbf{r},\tau)} \left[ \frac{1}{2} \mathbf{J}^2_{(\mathbf{r},\tau)} - (\tilde{\mu} + \tilde{v}_\mathbf{r}) J^\tau_{(\mathbf{r},\tau)} \right]. \tag{49}$$

Evidently, when $\tilde{v}_\mathbf{r} = 0$, integer values of $\tilde{\mu}$ can be absorbed into the definition of $J^\tau_{(\mathbf{r},\tau)}$, so the model reduces to the (2+1)D Villain model, which is in the same universality class as the (2+1)D XY model. These points, are, however, just special multicritical points and the generic behavior is *not* that of the XY model [1].

In order to describe the superconductor-insulator transition, it is neccesary to add long-range Coulomb forces. Although the precise form of a short range interaction does not change the universality class, long range interactions *are* relevant and so must be included. The time component, $J^\tau_{(\mathbf{r},\tau)}$, of the link variables corresponds to the particle density or boson occupation number so Coulomb interactions can be included in the following way:

$$H = H_V + H_c \tag{50}$$

$$H_c = \frac{e^{*2}}{K} \sum_\tau \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} (J^\tau_{(\mathbf{r},\tau)} - n_0) G(\mathbf{r} - \mathbf{r}') (J^\tau_{(\mathbf{r}',\tau)} - n_0). \tag{51}$$

Here $e^*$ is the effective boson charge, $n_0$, which represents the compensating background charge, is the average particle density, and $G$ is the Coulomb interaction. In the simulations with long-range interactions, the particle number was always kept constant, whereas when only short-range interactions were present the particle number was allowed to fluctuate.

## 10. FINITE SIZE SCALING

Near the critical point, the correlation length is much larger than the size of the system, so finite-size effects will be important. We therefore need to derive a finite-size scaling form [22] for the stiffness. The basic finite-size scaling hypothesis is that the size of the system only appears in the ratio $L/\xi$, and, for quantum problems, the corresponding ratio in the time direction, $L_\tau/\xi_\tau$. Thus we have

$$\rho(0) = \xi^{-(d+z-2)} P(L/\xi,\ L_\tau/\xi_\tau) \tag{52}$$

which can be more conveniently expressed as

$$\rho(0) = \frac{1}{L^{d+z-2}} \bar{\rho} \left( L^{1/\nu}\delta,\ \frac{L_\tau}{L^z} \right) , \tag{53}$$

where $P$ and $\bar{\rho}$ are scaling functions. It is thus essential to work with system shapes for which the aspect ratio

$$c = L_\tau / L^z \tag{54}$$

is a constant, otherwise the scaling function $\bar{\rho}$ depends on two variables and is complicated to analyze. If this is done, $L^{d+z-2}\rho$ is *independent* of $L$ at the critical point $\delta = 0$. Furthermore, in the disordered state, the system is insensitive to changes in the boundary conditions if the size is bigger than the correlation length, so $L^{d+z-2}\rho$ will *decrease* (exponentially) with increasing $L$. By contrast, in the ordered state, $\rho$ tends to a constant so $L^{d+z-2}\rho$ *increases* with increasing $L$. Thus, the critical point is located at the intersection of curves for $L^{d+z-2}\rho$ as a function of coupling $K$ for different lattice sizes. One can then determine $\nu$ from Eq. (53) by requiring that the data for different sizes (but fixed aspect ratio) collapse on top of each other in plot of $\rho(0)L^{d+z-2}$ against $L^{1/\nu}\delta$. Note that in order to choose the sample shapes in the simulation, we need to have already made a choice for $z$.

The conductivity is related to the frequency dependent stiffness which is given in terms of the current-current correlation function [20, 21, 23]

$$\rho(i\omega_n) = \frac{1}{L^2 L_\tau}[\langle | \sum_{(\mathbf{r},\tau)} e^{i\omega_n \tau} J^x_{(\mathbf{r},\tau)}|^2\rangle]_{av}, \tag{55}$$

by

$$\sigma(i\omega_n) = 2\pi G_Q \frac{\rho(i\omega_n)}{\omega_n} \; . \tag{56}$$

## 11. RESULTS

In order to determine universal properties at the Bose glass to superfluid transition, Monte Carlo simulations have been performed [20, 23] on the Hamiltonian, Eq. (50). First of all I will describe the results for the case of short range interactions. Then I will discuss the results when Coulomb, $(1/r)$, interactions are added, so the model should be in the correct universality class to describe the superconductor-insulator transition.

We start the analysis by locating the critical point. Since, as discsused above, scaling theory gives $z = d = 2$, the relevant quantity to plot, according to the finite-size scaling analysis above is $\rho(0)L^2$. Results for the aspect ratio of 1/4 results are shown in Fig. 4. Since the critical point is located where the curves cross, the figure demonstrates clearly that there is a transition close to $K = 0.25$ between a superfluid phase with finite superfluid density, $\rho_s$ (remember that $\rho_s \sim \rho(0)$), and an insulating phase with zero superfluid density. The best estimate of the critical coupling is $K_c = 0.248 \pm 0.002$. Close to the critical point 1000 to 2000 disorder realizations were performed, with, for the largest size, an equilibration time of $T_0 = 10,000$ followed by 10,000 MCS for averaging with a measurement every 10 MCS. Some runs were also done for sample shapes corresponding to other value of $z$. When $z$ was significantly different from two, the scaling was much less good. Fig. 5 shows the resistance per square as a function of frequency [24] evaluated from Eq. (56) for aspect ratio 1/4 at the critical point $K_c = 0.248$. For the two smallest

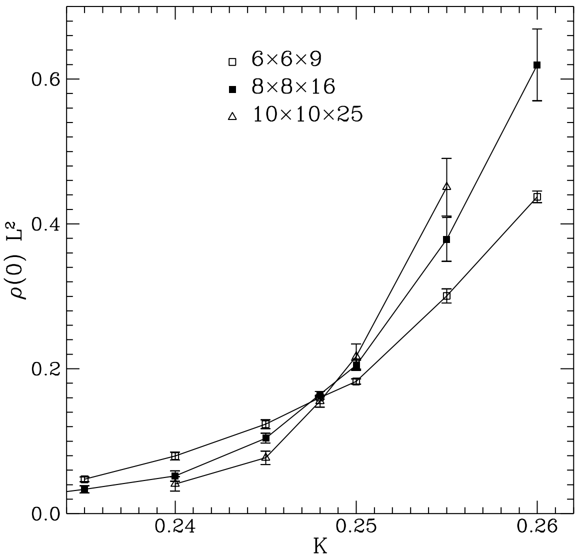


Figure 4. $\rho(0)L^2$ for the system sizes indicated, as a function of $K$ for short range interactions. From the intersection of the curves one estimates the critical coupling to be be $K_c = 0.248 \pm 0.00$. From Ref.[23].

system sizes about 2,000 disorder configurations were generated while for the largest size only 1,000 were done. From 3,000 to 10,000 MCS were done for equilibration followed by the same number of sweeps for measurements. To determine the universal conductivity we have to analytically continue the MC data to real frequency and extrapolate to $\omega = 0$. For typical quantum MC simulations, this analytic continuation is extremely difficult to perform. However it turns out to be straightforward in the present case, since the data for the resistivity varies linearly at small $\omega_n$, which implies,

$$\sigma(i\omega_n) = \frac{\sigma^*}{(1 + |\omega_n|\tau_0)} \ . \tag{57}$$

This is easily seen to analytically continue to the Drude form of the conductivity:

$$\sigma(\omega + i\delta) = \frac{\sigma^*}{(1 - i\omega\tau_0)} \ . \tag{58}$$

Here $\tau_0 \sim 1/\omega_c$ is a non-universal relaxation time.

The main source of error in the determination of the d.c. conductivity comes from the uncertainty in the determination of the critical point. This error was estimated by making the same linear fit to the resistivity data at the ends of the interval given by the error bars of the critical coupling. The final estimate for the critical conductivity is [20, 21, 23]

$$\sigma^* = (0.14 \pm 0.03)G_Q \ , \quad R^*_\square = (7.4 \pm 1.6)R_Q \ . \tag{59}$$

Estimates were also obtained for the various critical exponents, see Refs. [20, 23]. Recently, two different papers [26, 27] have obtained significantly different values for $\sigma^*$. The reason for this discrepancy is currently unclear.

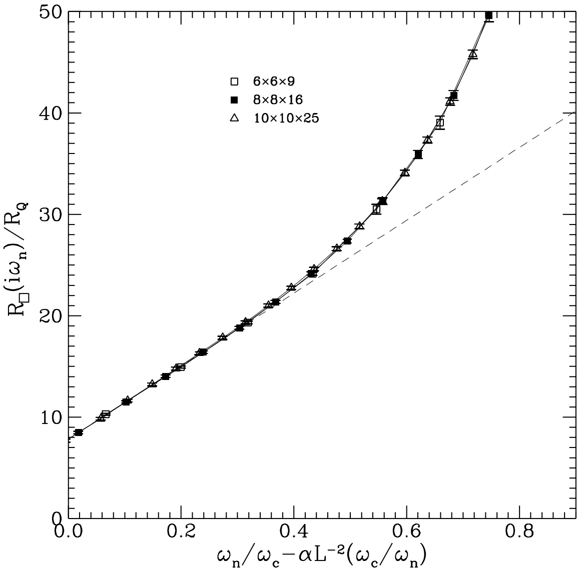


Figure 5. The resistivity in units of $R_Q = h/(2e)^2$, as a function of $\omega_n/\omega_c - \alpha L^{-2}(\omega_c/\omega_n)$ for the model with short range interactions, where $\omega_c = 2\pi$ and $\alpha = 0.179$. The calculation was done at the critical point, $K = 0.248$. The dashed line indicates a least square fit to the points with abscissa less than 0.26. The correction involving the parameter $\alpha$ is proportional to $T/\omega_n$. On physical grounds we expect that $\alpha = 0$, and indeed a fairly good fit, with almost the same value of the d.c. resistivity, is obtained with $\alpha = 0$. From Ref. [23].

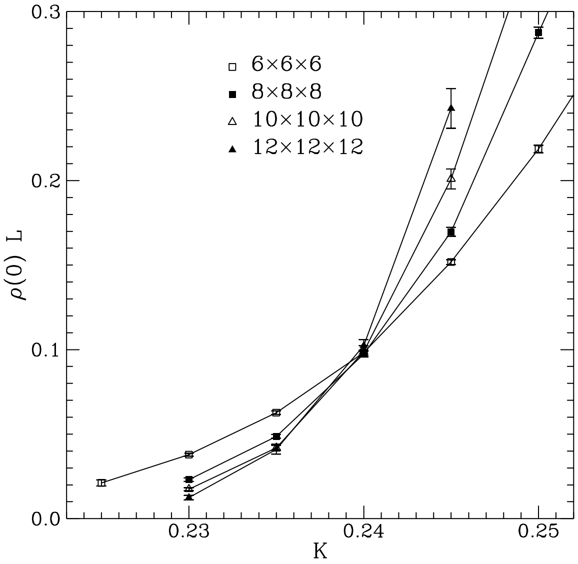


Figure 6. $\rho(0)L$ for the system sizes indicated, as a function of $K$ for long range interactionss with $e^{*2} = 1/2$. The critical point, determined from the intersections, is $K_c = 0.240 \pm 0.003$. From Ref. [23].

Next let us discuss results [20, 23] for long range Coulomb interactions. Scaling arguments have been presented [25] that $z = 1$, so it is natural to use systems with shape $L \times L \times L$ where we took $L = 6, 8, 10$ and 12. The number of disorder realizations varied from 200 to 1000. Since $z = 1$ it follows from the discussion after Eq. (54) that we should look for the intersections of data for $\rho(0)L$ against $K$. The results for $e^{*2} = 1/2$. are presented in Fig. 6. Clearly the lines cross close to $K = 0.240$ and, more precisely, we estimate the critical coupling to be $K_c = 0.240 \pm 0.003$, quite close to the value for the short range case. Since all four sizes intersect very close to the same point, Fig. 6 provides strong evidence that $z = 1$, in agreement with the scaling arguments.

Following the approach used above for the short range case, Fig. 7 shows the resistivity, $R_\square$ in units of $R_Q$, against frequency at the critial point, $K_c = 0.240$ with $e^{*2} = 1/2$. The collapse of the data is excellent. As for the short range case, the data varies linearly with $\omega_n$, implying a Drude form for the conductivity. A least squares fit to the data gave

$$R_\square^* = (1.82 \pm 0.20) R_Q, \quad \sigma^* = (0.55 \pm 0.06) G_Q. \tag{60}$$

This model should be in the same universality class as the supercondctor-insulator transition. Experiments [8–10], however, do not agree with each other as to the values of the critical conductivity, values of $\sigma^*/G_Q$ varying from about 0.6 to 2. However, it is possible that many of these experiments were not at sufficiently low temperature to probe the critical regime.

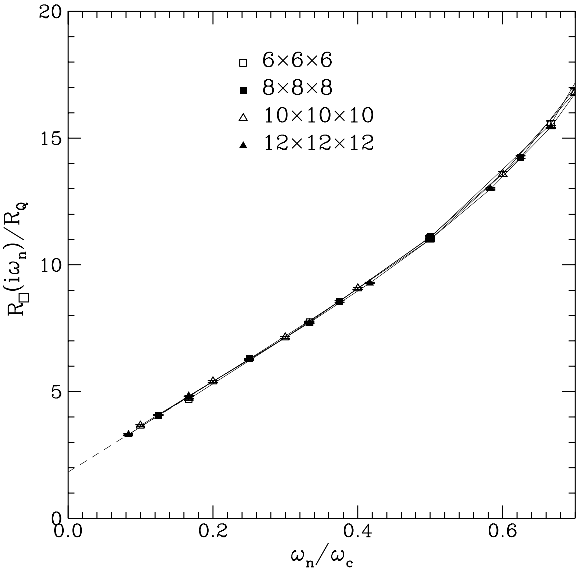


Figure 7. The resistivity in units of $R_Q = h/(2e)^2$, as a function of $\omega_n/\omega_c$ for the model with long range range interactions. The calculation was done at the critical point, $K = 0.240$. The dashed line indicates a least square fit to the points with abscissa less than 0.44. From Ref. [23].

## 12. CONCLUSIONS

In these lectures I have tried to emphasise the differences bewteen classical and quantum transitions. In the quantum case, theoretical work generally uses a path integral approach so one is effectively dealing with a $(d+1)$-dimensional problem, the extra dimension being imaginary time. Only in special cases, however, is the critical behavior that of the corresponding classical model in $(d+1)$ dimensions. More generally, there is a dynamical exponent, $z$, which modifies the classical hyperscaling relations by the replacement $d \to d+z$. If $z \neq 1$, the critical behavior is generally not that of any classical transition. In the presence of disorder, critical exponents cannot continuously approach their mean field values as the dimensionality tends to an upper critical dimension, $d_u$. Either there is *no* upper critical dimension or the exponents jump discontinuously to their mean field values at $d_u$. In either case, there is no $\epsilon$-expansion.

I have also illustrated the powerful technique of Monte Carlo simulations combined with finite-size scaling, which can make predictions for critical behavior even when, as here, analytical approaches fail.

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