Physics 5I
The Lorentz Transformation

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(Dated: October 13, 2009)

In this handout we show that the Lorentz transformation leads to the phenomena of time dilation, length contraction, and lack of simultaneity that we have discussed in class.

There will be two inertial frames:

- The "'track frame" of an observer standing (at rest) by the train tracks.
- The "train frame" of an observer in a freight car which moves with velocity v to the right along the tracks.

We indicate coordinates in the train frame by a prime and coordinates in the track frame without a prime. For example, the length of the car is L' according to the train observer and L according to the track observer.

Each observer will have initially synchronized a set of clocks by bringing them together, setting the times equal, and them slowly moving them to different points in his/her frame. Then, for example, if the track observer wants to know at what time the front of the train, say, reaches a certain point he will simply look at the time on his clock *at that point*.

The Lorentz transformation relates coordinates and times in the two frames by

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}\,(x - vt), \tag{1a}$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}\,(t - vx/c^2). \tag{1b}$$

The inverse transformation (relating x and t to x' and t') can be obtained either by solving these equations for x and t, or, more simply, by realizing that, as far as the train observer is concerned, the track is moving with speed v to the left, i.e. with *velocity* $-v$. Hence the inverse transformation is obtained simply by replacing v by $-v$, i.e.

$$x = \frac{1}{\sqrt{1 - v^2/c^2}}\,(x' + vt'), \tag{2a}$$

$$t = \frac{1}{\sqrt{1 - v^2/c^2}}\,(t' + vx'/c^2). \tag{2b}$$

We will need to compare the relative times and positions of *two* events "1" and "2". Defining $\Delta x = x_2 - x_1$, etc., the Lorentz transformations clearly have the same form for the Δx, Δt as they do for x, t, i.e.

$$\Delta x' = \frac{1}{\sqrt{1 - v^2/c^2}} (\Delta x - v\Delta t), \tag{3a}$$

$$\Delta t' = \frac{1}{\sqrt{1 - v^2/c^2}} (\Delta t - v\Delta x/c^2), \tag{3b}$$

$$\Delta x = \frac{1}{\sqrt{1 - v^2/c^2}} (\Delta x' + v\Delta t'), \tag{4a}$$

$$\Delta t = \frac{1}{\sqrt{1 - v^2/c^2}} (\Delta t' + v\Delta x'/c^2). \tag{4b}$$

Now we consider how the phenomena of time dilation, length contraction, and lack of simultaneity are described by the Lorentz transformation.

A. Time dilation

The track observer observes a clock (at a *fixed position*) on the train, so $\Delta x' = 0$. Equation (4b) gives $\Delta t = \Delta t'/\sqrt{1 - v^2/c^2}$, so

$$\boxed{\Delta t' = \Delta t\sqrt{1 - v^2/c^2} \qquad (\text{with}\, \Delta x' = 0).} \tag{5}$$

In other words, less time has passed for the train observer than for the track observer.

Note that we are comparing measurements of time with *one* clock in the train frame (where it is stationary) to measurements with *two* clocks at different points in space in the track frame. According to Eq. (4a), the track observer observes the train clock to move a distance $\Delta x = v\Delta t'/\sqrt{1 - v^2/c^2}$ between the two events, and so he will determine Δt from times on clocks *separated by this distance.*

Of course, we could equally well argue that, to the train observer, clocks on the track run slow. However, we are then comparing *one* clock in the track frame (where it is stationary so we have $\Delta x = 0$ rather than $\Delta x' = 0$) with *two* in the train frame. Since clocks synchronized in one frame are not synchronized in the other (see below), there is no contradiction in the two different measurements coming to different conclusions as to which set of clocks run slower.

B. Length contraction

Suppose we measure the length of an object in the freight car. The train observer determines its length to be $\Delta x'$. The track observer will measure the distance apart of the two ends of the object measured *at the same time* in his frame, so $\Delta t = 0$. Equation (3a) then gives $\Delta x' = \Delta x/\sqrt{1 - v^2/c^2}$, so the length of the object is given by

$$\boxed{\Delta x = \Delta x' \sqrt{1 - v^2/c^2} \qquad (\text{with}\, \Delta t = 0),} \tag{6}$$

for the track observer. In other words, to the track observer, the object appears shorter than it does to an observer at rest with respect to it.

Again, the observer in the train could claim that it is really the other way round, namely objects at rest in the track frame appear to have shrunk. And again, as for time dilation, the resolution of this paradox is that different measurements are being done, and observers in different frames disagree about whether clocks are synchronized (and so disagree as to whether two observations at different points in space are done *at the same time*).

C. Lack of simultaneity

Here we consider events at two points in space which occur at the same time in the train frame, so $\Delta t' = 0$. According to Eq. (3b), for the track observer they occur at times which differ by

$$\boxed{\Delta t = \frac{v}{c^2}\, \Delta x \qquad (\text{with}\, \Delta t' = 0)\,.} \tag{7}$$

In class we considered a light beam sent from the center of car in both directions and the two events were the light beam striking each end. We wrote an expression for the lack of synchronization for the track observer in terms of the length L of the car, as measured by him, rather than Δx, the distance apart in the track frame of the points where the light struck the two ends. The latter is equal to the length of the car, L, plus v times the time difference Δt between the two events, i.e.

$$\Delta x = L + v \left[\frac{L/2}{c - v} - \frac{L/2}{c + v} \right] = \frac{L}{1 - v^2/c^2} \,, \tag{8}$$

so Eq. (7) can also be written

$$\Delta t = \frac{v}{c^2}\, \frac{L}{1 - v^2/c^2} \,, \tag{9}$$

which is the expression derived in class.