PHYSICS 250
Homework 6
Due in class, Monday November 12.

1. **Fourier Transform of a Bessel function**

   Consider the Bessel function $J_0(x)$ for positive and negative x (recall that $J_0(x)$ is an even function). Use the integral representation

   $$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ix\cos\theta)\, d\theta \tag{1}$$

   to show that the Fourier transform can be expressed as

   $$g(k) = \int_{-\infty}^{\infty} e^{ikx}\, J_0(x)\, dx = \int_0^{2\pi} \delta(k + \cos\theta)\, d\theta.$$

   Noting that the delta function is never satisfied for $|k| > 1$, and that there are two values of the θ which satisfy it for $|k| < 1$, show that

   $$g(k) = \begin{cases} \dfrac{2}{\sqrt{1-k^2}}, & (|k| < 1) \\ 0, & (|k| > 1) \end{cases}$$

2. **Discrete Fourier Transform**
   For this question you should create a Mathematica notebook. Please ask me if you have any problems with this.

   (a) Consider a time series consisting of values of the Bessel function $J_0(x)$. Assume that the data covers a range from $-X/2 + \delta x \le x \le X/2$ in intervals of δx. Hence the total range is X and the number of points is $N = X/\delta x$.
   Consider, for example, $X = 200, \delta x = 0.2$.
   Use Mathematica to plot the data. Note that the data has not fallen to zero at the end points.

   (b) Compute (using the Mathematica function `Fourier`) the power spectrum and plot it. Note the (unphysical) oscillating behavior which comes because the function represented by the Fourier Transform is a periodic continuation of the time series, and this has discontinuities every X.

   (c) On the same plot show the power spectrum obtained from the time series and the exact power spectrum evaluated in the continuum with the time range extended to $\pm\infty$ (i.e. the square of the result in Qu. (2)).

   (d) Multiply the data by the Welch windowing function discussed in class, and again plot the power spectrum from the data in the same figure as the exact result. Show that the two now agree very well, and the oscillating behavior in the numerical data, coming from (the discontinuity at) the end points, is now absent.

3. (a) Using the integral representation for the Bessel function $J_0(x)$ in Eq. (1), determine the Laplace transform of $J_0(x)$.

   (b) Perform the inverse Laplace transform and show that you get the integral representation in Eq. (1) (for $x > 0$).
   Hint: Put the branch cut to run between $k = i$ and $-i$ and the answer will come from the discontinuity across the cut.

4. Use Laplace transforms to solve the following ordinary differential equations in the range $t > 0$:

   (a)
$$\frac{dx}{dt} - Kx = e^{Kt}\,, \qquad x(0) = 0\,.$$

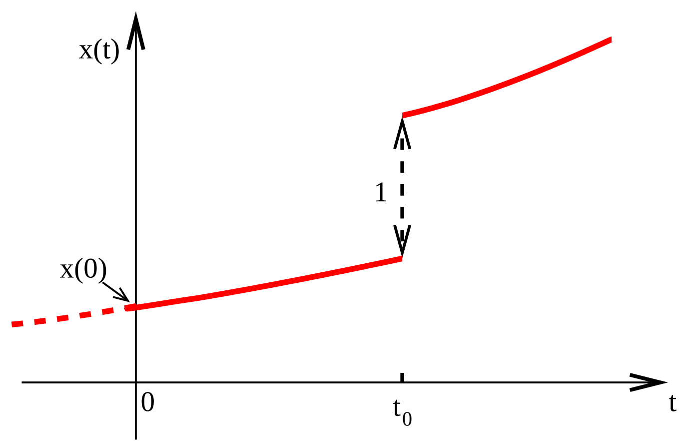
   (b)
$$\frac{d^2x}{dt^2} + 9x = \sin(2t)\,, \qquad x'(0) = x(0) = 0\,.$$

5. A comparison of Fourier and Laplace transforms (based on a suggestion of O. Narayan). Consider the following ordinary differential equation
$$\frac{dx(t)}{dt} = Kx + \delta(t - t_0)\,, \tag{2}$$
with $K > 0$ and $t_0 > 0$.

   (a) Show (this is trivial) that without the delta function the solution of Eq. (2) is
$$x(t) = x(0)e^{Kt}\,,$$
   which, since $K > 0$, grows exponentially with time.

   (b) The delta function gives an additional "kick" at $t = t_0$. To be precise show that the delta function causes x to increase by 1 at $t = t_0$, *i.e.*
$$\lim_{\delta \to 0} \left[x(t_0 + \delta) - x(t_0 - \delta)\right] = 1\,.$$

   (c) Show, using Laplace transforms, that the solution of Eq. (2) for $t > 0$ is
$$\begin{aligned} x(t) &= x(0)e^{Kt} + \theta(t - t_0)e^{K(t-t_0)} \\ &= \left[x(0)e^{Kt_0} + \theta(t - t_0)\right] e^{K(t-t_0)}\,, \end{aligned} \tag{3}$$
   where $\theta(x)$ is the step function which has values $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. The solution in Eq. (3) is plotted in the figure below for a certain value of $x(0)$.



   Note:

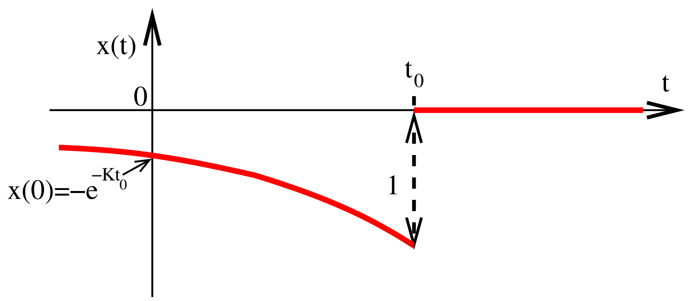
   i. You may use the following results for inverse Laplace transforms:
$$\mathcal{L}^{-1}\left(\frac{1}{s-K}\right) = e^{Kt}\,, \quad \mathcal{L}^{-1}\left(\frac{e^{-st_0}}{s-K}\right) = \theta(t - t_0)e^{K(t-t_0)}\,,$$
   which you can also either derive from the appropriate contour integral or verify by working out the Laplace transforms on the right hand side.

ii. The solution from Laplace transforms, Eq. (3), is obtained for $t > 0$. However, one can continue that solution to negative times if one wishes, as shown by the dashed line in the above figure.

(d) Now solve the problem (for all x) using Fourier transforms. Show that this gives

$$x(t) = [-1 + \theta(t - t_0)]\, e^{K(t-t_0)}\,,$$

which *vanishes* for $t > t_0$. This is shown in the figure below.



Note that the parameter $x(0)$ does not appear (boundary conditions at the origin do not appear in Fourier transforms[1]. Rather, the boundary conditions which are *automatically* imposed in Fourier transforms is that the solution vanishes at $\pm\infty$. To prevent the solution diverging for $x \to \infty$ the Fourier transform method automatically picks the value of $x(0)$ in Eq. (3) to be $-e^{-Kt_0}$ so that $x(0)e^{Kt_0}$ in Eq. (3) is equal to -1. Hence the $x(0)e^{Kt}$ solution is "killed", for $t > t_0$, by the delta function "kick" at t_0.

Note: the solution vanishes automatically for $t \to -\infty$ for *any* $x(0)$ since $K > 0$.

[1] This is not quite true; they do appear in sine and cosine transforms. However, sine and cosine transforms are only useful for *second order* equations, because the *sine* transform of the *first* derivative is related to the *cosine* transform and vice-versa, so sine- or cosine-transforming a first derivative does not help solve the problem. They *can* be used for second order equations, (because the *sine* transform of the second derivative is related to the *sine* transform of the function, and similarly for the cosine transform), but for these problems only *one* of the two boundary conditions is specified at the origin, $f(0)$ for the sine transform and $f'(0)$ for the cosine transform, and the *other* boundary condition is that the solution vanishes at $+\infty$. With Laplace transforms *all* the boundary conditions are specified at the origin, and the solution *may* diverge at $+\infty$.