PHYSICS 250
Homework 5
Due in class, Monday November 5

1. Show that
$$\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}\,d\omega}{E_0 - i\Gamma/2 - \hbar\omega} = \begin{cases} \exp(-\Gamma t/2\hbar)\exp(-iE_0 t/\hbar), & (t>0), \\ 0, & (t<0), \end{cases}$$
for $\Gamma > 0$. This time Fourier transform appears in quantum mechanics.

2. In quantum mechanics the wave function $\psi(x)$ has the property that $P(x) \equiv |\psi(x)|^2$ is the probability density for the particle, i.e. the probability that it lies between x and $x + dx$ is $|\psi(x)|^2\,dx$. Hence, for example, the mean position is given by $\langle x\rangle = \int xP(x)\,dx$. We define the uncertainty in position by
$$(\Delta x)^2 = \left\langle\, (x - \langle x\rangle)^2 \,\right\rangle = \langle x^2\rangle - \langle x\rangle^2\ .$$
One can show that the "momentum wavefunction" $g(p)$ is given by
$$g(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x)e^{-ipx/\hbar}\,dx\,,$$
i.e. $g(p)$ is (apart from the factor of $\hbar$) the Fourier transform of $f(x)$. This connection is one of the important applications of Fourier transforms in physics. Now $g(p)$ has the property that $|g(p)|^2 dp$ is the probability that the momentum lies betwen p and $p + dp$.
   (a) Using Parseval's theorem, show that if $\psi(x)$ is normalized, i.e. if $\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1$, then $\int_{-\infty}^{\infty}|g(p)|^2\,dp = 1$, so the momentum wave function is also normalized.
   (b) Consider the Gaussian wavepacket
$$\psi(x) = \frac{1}{\pi^{1/4}a^{1/2}}\exp\left(-\frac{x^2}{2a^2}\right)\ .$$
      i. Show that it is correctly normalized.
      ii. Determine Δx.
      iii. Determine $g(p)$ and hence Δp.
      iv. Hence show that
$$\Delta x\Delta p = \frac{\hbar}{2}\ .$$
      This is, of course, an example of Heisenberg's famous uncertainty principle.
      Note: One can show that for *any* $\psi(x)$, the product $\Delta x\Delta p$ can not be less than this, and hence a Gaussian is a "minimum uncertainty" wavepacket.

3. Determine the Fourier transform of
$$f(x) = \begin{cases} 1, & (|x| < a) \\ 0, & (|x| > a) \end{cases}\ .$$
Hence, using the convolution theorem, evaluate
$$\int_{-\infty}^{\infty}\frac{\sin ak\,\sin bk}{k^2}\,dk\,,$$
assuming that both a and b are positive.
Note: You may need to consider the two possibilities, $a > b$ and $b > a$.

4. As an example of a singular Fourier transform, we determine here the Fourier transform of $f(x) = x$, *i.e.* determine $g(k)$ where

$$g(k) = \int_{-\infty}^{\infty} x e^{ikx} \, dx \,,$$

with the usual caveat that this is understood *either* to have a convergence factor $e^{-\epsilon|x|}$ in the integrand *or* both sides are to be multiplied by a smooth function of k and integrated.

   (a) Show that $g(k) = -iG'(k)$ where $G(k)$ is the Fourier transform of $F(x) = 1$. Since we showed in class that $G(k) = 2\pi\delta(k)$ this gives

$$g(k) = -2\pi i \delta'(k) \,.$$

   (b) Show that the inverse transformation of $g(k)$ correctly gives $f(x) = x$.

5. Consider neutrons diffusing in graphite. The density of neutrons $n(x,t)$, assumed to vary only in the x-direction, satisfies the diffusion equation

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} \,.$$

Now assume that at time $t = 0$, Q neutrons are suddenly placed at the origin, i.e.

$$n(x,0) = Q\,\delta(x) \,.$$

By Fourier transforming with respect to x show that

$$n(x,t) = Q \, \frac{e^{-x^2/4Dt}}{\sqrt{4\pi D t}} \,.$$

Note: This shows that the neutrons spread out (as expected), and, at fixed time, the density has a Gaussian dependence on x with

$$\begin{aligned} \langle x(t) \rangle \equiv \int_{-\infty}^{\infty} x\, n(x,t)\, dx &= 0 \\ \langle x(t)^2 \rangle \equiv \int_{-\infty}^{\infty} x^2\, n(x,t)\, dx &= 2Dt \,. \end{aligned}$$

Comment:

   (a) $\langle x(t) \rangle = 0$ means that the neutrons spread out equally in both positive and negative directions.

   (b) $\langle x(t)^2 \rangle = 2Dt$ means that the characteristic distance moved after a time t is proportional to $t^{1/2}$. This $t^{1/2}$ behavior is characterstic of diffusion. It differs from propagation of waves, or collisionless propagation of particles (ballistic propagation), where the distance traveled is proportional to t.

6. Consider the temperature $u(x,t)$ of a semi-infinite bar in the region $0 < x < \infty$. It satisfies the diffusion equation:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \,.$$

At negative times the temperature is T_0 for $0 < x < 1$ and 0 for $x > 1$. At $t = 0$, the end at $x = 0$ is put at $u = 0$ and no heat is allowed to leave the sides of the bar. Using a sine Fourier transform with respect to x, find the temperature $u(x,t)$ for $t > 0$ and $x > 0$.

Note: The last integral is not quite trivial. To determine it, first form $\partial u(x,t)/\partial x$, then carry out the integral over k, and finally integrate the result with respect to x determining the arbitrary constant from the requirement that $u(0,t) = 0$.

You use the sine Fourier transform to ensure that $u(0,t) = 0$. In class we showed that the sine Fourier transform of $f''(x)$ is $-k^2 g_s(k)$ (where $g_s(k)$ is the sine Fourier transform of $f(x)$), provided $f(x)$ vanishes at $x = 0$ (the situation here).