Physics 250
Bessel Functions, Asymptotic Expansions and Stokes' Phenomenon

Peter Young

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Contents

I. INTRODUCTION

Bessel functions occur in many areas of physics and mathematics. Examples are Laplace's equation in two dimensions, and the three-dimensional Helmholtz equation, $(\nabla^2 + k^2)\psi = 0$, which comes from separating variables in the Schrödinger equation, the diffusion equation and the wave equation. They are probably the most extensively studied of the special functions. In these notes we discuss the contour integral representation from which we will derive asymptotic expansions, valid for large $|z|$, and discuss the range of values of $\arg z$ for which these expansions are valid.

II. DIFFERENTIAL EQUATION

The solutions of the differential equation

$$z^2 f'' + z f' + (z^2 - \nu^2) f = 0 \tag{1}$$

are called the Bessel functions of order ν. We will take ν to be real. If we are discussing results which are only true if ν is an integer, we will generally denote it by n. z will denote a complex variable. In special cases where we require it to be real we will generally denote it by x.

III. SERIES SOLUTION FOR J_ν

A solution of Eq. (1) is

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \left[\sum_{k=0}^{\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k!(\nu+k)!}\right], \tag{2}$$

where $(\nu + k)! \equiv \Gamma(\nu + k + 1)$, which is called the *Bessel function of the first kind.* Eq. (2) is easily obtained by taking a trial solution of the form $f(z) = z^\alpha \sum_{k=0}^\infty a_k z^k$, with $a_0 \neq 0$, and substituting into Eq. (1). One finds that $\alpha = \nu$ (or $-\nu$) and one easily determines the ratio of the coefficients, which leads to Eq. (2). Note that $J_\nu(x)$ is real if x is real and positive.

From the ratio test, it is easy to see that the series has infinite radius of convergence, so Eq. (2) can be taken as a *definition* of $J_\nu(z)$. It also follows that $J_\nu(z)$ is an analytic function everywhere in the (finite) complex plane except that, unless ν is an integer, there is a branch cut starting at the origin, because of the factor z^ν. We shall consider the principal branch, obtained for

$$-\pi < \arg z \le \pi, \tag{3}$$

so the branch cut for $J_\nu(z)$, as well as for the other functions that we will discuss, goes along the -ve real axis. From Eq. (2) we see that

$$J_\nu(e^{im\pi} z) = e^{im\nu\pi} J_\nu(z), \tag{4}$$

for integer m, since $(z^2/4)^k$ in Eq. (2) is unchanged under the transformation $z \to e^{im\pi} z$.

Both $J_\nu(z)$ and $J_{-\nu}(z)$ are solutions of Eq. (1). If ν is not an integer then these two functions are linearly independent (which is obvious because the powers of z in the two functions are different). Hence the general solution for ν non-integer is

$$f(z) = AJ_\nu(z) + BJ_{-\nu}(z). \tag{5}$$

IV. BESSEL FUNCTIONS OF INTEGER ORDER

If ν is a positive integer, n say, then

$$J_n(z) = \left(\frac{z}{2}\right)^n \left[\frac{1}{0!n!} - \left(\frac{z}{2}\right)^2 \frac{1}{1!(n+1)!} + \left(\frac{z}{2}\right)^4 \frac{1}{2!(n+2)!} + \cdots\right]. \tag{6}$$

Note that

$$J_n(-z) = (-1)^n J_n(z). \tag{7}$$

The Bessel functions, $J_n(x)$, for $n = 0, 1, 2$ and 3 are shown in Fig. 1. One sees that they oscillate, with a phase which changes with n, and gradually decay. The precise form of this oscillation and decay will be elucidated later in §IX.

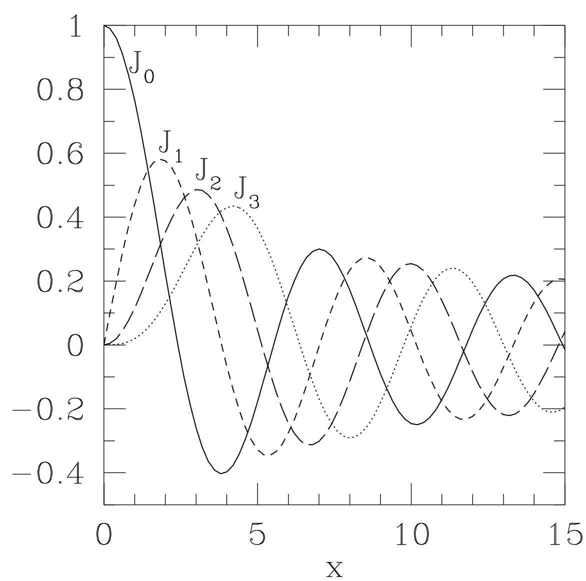

FIG. 1. A plot of the Bessel functions $J_n(x)$ with $n = 0, 1, 2$ and 3.

It is sometimes useful to study the generating function for Bessel functions of integer order. This is given by

$$e^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z), \tag{8}$$

which will be derived is §VIII E below when we discuss the integral representation of Bessel functions.

If ν is a negative integer, $-n$ say, then, since $1/m! = 0$ for m a negative integer, the first n terms in Eq. (2) are all zero, and the leading term is of order z^n. It is therefore easy to see that

$$J_{-n}(z) = (-1)^n J_n(z). \tag{9}$$

From Eq. (9), it follows $J_n(z)$ and $J_{-n}(z)$ are not independent, so we need to find a second solution of Eq. (1) for ν integer. It turns out to be useful to consider the function

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \tag{10}$$

which is called the *Bessel function of the second kind.* (Sometimes it is called the Weber function or Neumann function and it is quite often denoted by $N_\nu(z)$). If ν is not an integer, so $\sin\nu\pi \neq 0$, then Y_ν is just a linear combination of J_ν and $J_{-\nu}$. However, if one lets ν approach an integer, n, and defines Y_n by this limiting value, then it turns out that $Y_n(z)$ is the desired second solution. Since $Y_n(z)$ can not be obtained by the series solution method, it is of no surprise that it does not have a regular series expansion about the origin. It turns out that there are terms involving $\ln z$ so $\lim_{z\to 0} Y_n(z) = \infty$. Note that $Y_\nu(x)$ is real if x is real and positive.

V. HANKEL FUNCTIONS, $H_\nu^{(1)}$ AND $H_\nu^{(2)}$

It is also convenient to define

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \qquad = \frac{i}{\sin\nu\pi}\left\{e^{-i\nu\pi}J_\nu(z) - J_{-\nu}(z)\right\} \tag{11}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \qquad = \frac{i}{\sin\nu\pi}\left\{J_{-\nu}(z) - e^{i\nu\pi}J_\nu(z)\right\}, \tag{12}$$

which are called *Hankel functions* or *Bessel functions of the third kind.* They are useful because of their special behavior at large $|z|$, see Eqs. (85) and (86) below. Note also that

$$J_\nu(z) = \tfrac{1}{2}\left(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right), \tag{13}$$

and

$$H_{-\nu}^{(1)}(z) = e^{i\pi\nu}H_\nu^{(1)}(z) \tag{14}$$

$$H_{-\nu}^{(2)}(z) = e^{-i\pi\nu}H_\nu^{(2)}(z) \tag{15}$$

$$H_\nu^{(1)}(e^{im\pi}z) = -e^{-im\nu\pi}H_\nu^{(2)}(z) \tag{16}$$

$$H_\nu^{(2)}(e^{-im\pi}z) = -e^{im\nu\pi}H_\nu^{(1)}(z), \tag{17}$$

for integer m. In addition, if x is real and positive, we have

$$H^{(1)}(x) = H^{(2)}(x)^\star. \tag{18}$$

VI. MODIFIED BESSEL FUNCTIONS, I_ν AND K_ν

It is frequently necessary to discuss Bessel functions with imaginary argument. These are solutions of the equation

$$z^2 f'' + z f' - (z^2 + \nu^2) f = 0. \tag{19}$$

They occur sufficiently often that they are generally expressed as new functions. A solution of Eq. (19) is $I_\nu(z)$, which has a series expansion similar to Eq. (2) except that all the coefficients are positive, *i.e.*

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \left[\sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!(\nu+k)!}\right]. \tag{20}$$

This series, which has an infinite radius of convergence, can be taken as the *definition* of $I_\nu(k)$, which is called a *modified Bessel function of the first kind.* Note that $I_\nu(x)$ is real if x is real and positive. $I_\nu(z)$ is analytic in the (finite) complex plane except that (unless ν is an integer) there is a cut along the negative real axis. (Since $I_\nu(z)$ is related to $J_\nu(iz)$ this does *not* correspond to the position of the cut for $J_\nu(z)$.) Hence one has

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu(e^{i\pi/2} z) \qquad (-\pi < \arg z \le \tfrac{\pi}{2}) \tag{21}$$

$$= e^{3i\nu\pi/2} J_\nu(e^{-i3\pi/2} z) \qquad (\tfrac{\pi}{2} < \arg z \le \pi), \tag{22}$$

where different definitions are needed in different sectors of the complex plane because the branch cut of $I_\nu(z)$ is not in the same place as the branch cut of $J_\nu(iz)$. Note that $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent if ν is not an integer, but they are equal, (see Eq. (26) below) if ν is an integer.

The generating function for modified Bessel functions of integer order is given by

$$e^{\frac{z}{2}(t+1/t)} = \sum_{n=-\infty}^{\infty} t^n I_n(z), \tag{23}$$

which will be derived in the §VIII E.

It is also convenient to define a second function $K_\nu(z)$ by

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \tag{24}$$

known as a *modified Bessel function of the third kind*, because it closely related to a Hankel function, see Eq. (25) below. *n.b.* The nomenclature is not standardized; sometimes this function is called the modified Bessel function of the second kind, sometimes the Macdonald function and sometimes the Basset function. Fortunately, the symbol $K_\nu(z)$, if not the name, is standard. For $\nu = n$, an integer, we define $K_n(z)$ by $\lim_{\nu \to n} K_\nu(z)$, and then $K_n(z)$ is the second solution of of Eq. (19), linearly independent of $I_\nu(z)$. $K_\nu(z)$ can be expressed in terms of a Hankel function as follows:

$$K_\nu(z) = \tfrac{i\pi}{2} e^{\frac{i\pi\nu}{2}} H_\nu^{(1)}(z e^{\frac{i\pi}{2}}) \qquad (-\pi < \arg z \le \tfrac{\pi}{2})$$
$$= -\tfrac{i\pi}{2} e^{\frac{-i\pi\nu}{2}} H_\nu^{(2)}(z e^{\frac{-i\pi}{2}}) \qquad (-\tfrac{\pi}{2} < \arg z \le \pi). \tag{25}$$

The domain of validity of these expressions comes from the locations of the branch cuts of the functions involved. Note also the following results:

$$I_{-n}(z) = I_n(z) \tag{26}$$
$$K_{-\nu}(z) = K_\nu(z) \tag{27}$$
$$I_\nu(e^{im\pi} z) = e^{im\nu\pi} I_\nu(z), \tag{28}$$

where m is an integer. There is not a simple connection between $K_\nu(z)$ and $K_\nu(e^{im\pi} z)$.

Even when ν is not an integer, it is often still convenient to take $K_\nu(z)$ and $I_\nu(z)$ as the independent solutions, rather than $I_{-\nu}(z)$ and $I_\nu(z)$, because, as indicated in Eqs. (93) and (98) below, $K_\nu(z)$ is the only linear combination of $I_{-\nu}(z)$ and $I_\nu(z)$ which *decreases* exponentially for $\mathrm{Re}(z) > 0$, rather than *increasing* exponentially.

VII. BESSEL FUNCTIONS OF HALF-INTEGER ORDER

Bessel functions whose order is a half-integer, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$ play an important role in mathematical physics, because they arise in solutions of Helmholtz's equation in three dimensions. One conventionally defines a spherical Bessel function of the first kind by

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z), \tag{29}$$

a spherical Bessel function of the second kind by

$$y_n(z) = \sqrt{\frac{\pi}{2z}} Y_{n+\frac{1}{2}}(z), \tag{30}$$

and spherical Bessel functions of the third kind (spherical Hankel functions) by

$$h_n^{(1)}(z) = j_n(z) + i y_n(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(1)}(z) \tag{31}$$
$$h_n^{(2)}(z) = j_n(z) - i y_n(z) = \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(2)}(z). \tag{32}$$

Spherical Bessel functions can be expressed in terms of a *finite* number of trigonometric functions. For example, by comparing series expansions of the two sides one can show that

$$j_0(z) = \frac{\sin z}{z}, \qquad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \tag{33}$$
$$y_0(z) = \frac{\cos z}{z}, \qquad y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}. \tag{34}$$

More complicated expressions apply for higher order.

One can also define modified spherical Bessel functions by

$$i_n(z) = \sqrt{\frac{\pi}{2z}} I_{n+\frac{1}{2}}(z) \tag{35}$$

$$k_n(z) = \sqrt{\frac{2}{\pi z}} K_{n+\frac{1}{2}}(z), \tag{36}$$

so, for example, we have

$$i_0(z) = \frac{\sinh z}{z}, \qquad i_1(z) = \frac{\cosh z}{z} - \frac{\sinh z}{z^2}, \tag{37}$$

$$k_0(z) = \frac{e^{-z}}{z}, \qquad k_1(z) = e^{-z}\left(\frac{1}{z} + \frac{1}{z^2}\right). \tag{38}$$

VIII. REPRESENTATION AS A CONTOUR INTEGRAL

A. Contour Integral for J_ν

Many useful results on Bessel functions can be derived from their representation as a contour integral. We shall verify that

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) dt, \tag{39}$$

where the path C encircles the branch cut of $t^{-(\nu+1)}$ anti-clockwise and $\arg t \to \pm\pi$ as the path goes to infinity, see Fig. 2. As we shall see, all we really need is that the integrand vanishes at the ends of the path (or has the same value at each end), so we only need the less restrictive condition that, when the path tends to infinity, $\pi \geq \arg z > \pi/2$ on the upper part and $-\pi < \arg z < -\pi/2$ on the lower part.

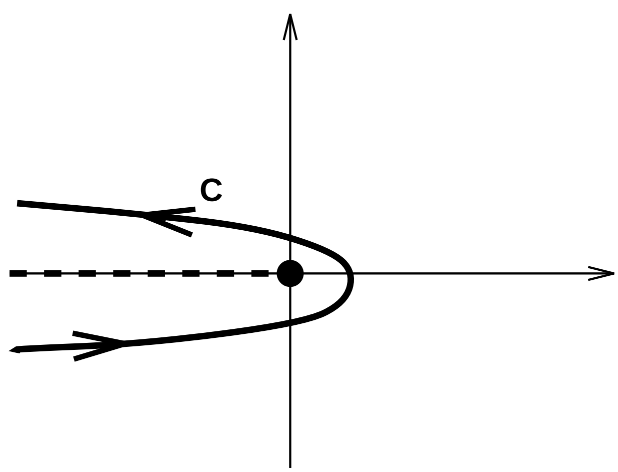

FIG. 2. The path, C, in the complex t-plane for the integral representation of $J_\nu(z)$ in Eq. (39). The path must go round the branch cut (indicated by the dashed line) and the pole (indicated by the solid circle) in an anti-clockwise sense. As the path goes off to infinity the argument tends to π on the upper part and $-\pi$ on the lower part. .

With the replacement $t \to zt/2$, one can replace Eq. (39) by

$$J_\nu(z) = \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] dt, \tag{40}$$

for $|\arg z| < \pi/2$, with the same contour C as in Fig. 2.

B. Series Expansion

We will now verify that the function given by Eq. (39) has precisely the same series expansion as Eq, (2) and so is the same function. Expanding the integrand in powers of z one has

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \left(-\frac{z^2}{4}\right)^k \frac{1}{k!}\frac{1}{2\pi i}\int_C \frac{e^t}{t^{\nu+k+1}}\, dt. \tag{41}$$

Now

$$\frac{1}{2\pi i}\int_C \frac{e^t}{t^{\nu+k+1}}\, dt = \frac{1}{\Gamma(\nu+k+1)} \equiv \frac{1}{(\nu+k)!}, \tag{42}$$

since this is the Hankel definition of the Γ function, see *e.g.* Carrier *et al.* Eq. (5-11), so we recover Eq. (2).

C. Differential Equation

We will now verify that the function given by Eq. (39) satisfies the differential equation, Eq. (1). Differentiating Eq. (39) under the integral sign we have

$$\frac{d^2 J_\nu}{dz^2} + \frac{1}{z}\frac{dJ_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) J_\nu = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}}\left\{1 - \frac{\nu+1}{t} + \frac{z^2}{4t^2}\right\} \exp\left(t - \frac{z^2}{4t}\right) dt \tag{43}$$

$$= \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_C \frac{d}{dt}\left[\frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right)\right] dt, \tag{44}$$

$$= 0, \tag{45}$$

since the integrand vanishes at the end points.

D. Contour Integral for $I_\nu(z)$.

It is easy to derive, along the same lines, a contour integral representation for $I_\nu(z)$. The result is

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}} \exp\left(t + \frac{z^2}{4t}\right) dt, \tag{46}$$

where C is the path in Fig. 2 and, for $|\arg z| < \pi/2$, this can be expressed as

$$I_\nu(z) = \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}} \exp\left[\frac{z}{2}\left(t + \frac{1}{t}\right)\right] dt. \tag{47}$$

E. Generating Function for Integer Order

From the integral representation in Eq. (40) we will now derive the generating function for Bessel functions of integer order in Eq. (8). For $\nu = n$, an integer, Eq. (40) is

$$J_n(z) = \frac{1}{2\pi i} \oint_C \frac{1}{t^{n+1}} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] dt. \tag{48}$$

where, since n is an integer, there is no branch cut and hence the path in Eq. (40) can be replaced by any closed loop encircling the origin anti-clockwise, as shown in Fig. 3. Eq. (45) is still valid because the integrand returns to its original value after circling the contour.

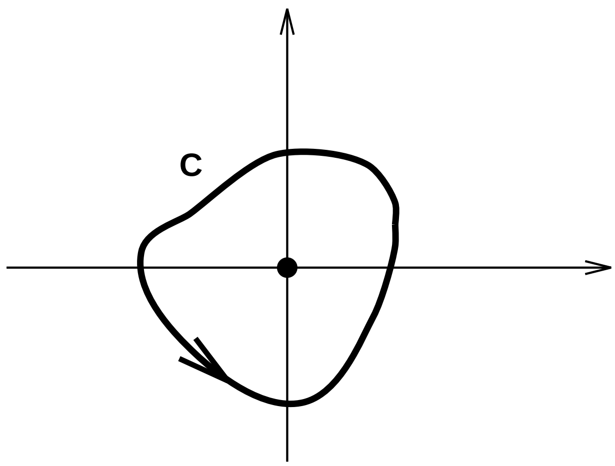

FIG. 3. The contour for the integral representation of $J_n(z)$ in Eq. (48), where n is an integer.

From the residue theorem, it follows that $J_n(z)$ is just the coefficient of t^n in the expansion of $\exp\left[\frac{z}{2}(t - 1/t)\right]$, *i.e.* one recovers, Eq. (8).

Similarly, using the integral representation of the modified Bessel function, $I_\nu(z)$, in Eq. (47) one obtains the generating function in Eq. (23).

F. Real Integral Representations for Integer Order

Let us assume that the order, n, is integer and make the contour in Fig. 3 a circle of unit radius. Writing $t = e^{i\theta}$, Eq. (48) becomes

$$J_n(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{e^{i(n+1)\theta}} i e^{i\theta} \exp\left[\frac{z}{2} 2i \sin\theta\right] d\theta \tag{49}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} e^{iz\sin\theta}\, d\theta. \tag{50}$$

Splitting the interval from 0 to 2π into an interval from 0 to π plus an interval from π to 2π, and making the replacement $\theta \to 2\pi - \theta$ in the latter, we have

$$J_n(z) = \frac{1}{2\pi} \int_0^{\pi} e^{-in\theta} e^{iz\sin\theta}\, d\theta + \frac{1}{2\pi} \int_0^{\pi} e^{in\theta} e^{-iz\sin\theta}\, d\theta \tag{51}$$

$$= \frac{1}{\pi} \int_0^{\pi} \cos\left(n\theta - z\sin\theta\right)\, d\theta. \tag{52}$$

This is a convenient expression for the Bessel function of integer order.

We can also get a similar expression for $I_n(z)$. For $\nu = n$, an integer, Eq. (47) is

$$I_n(z) = \frac{1}{2\pi i} \oint_C \frac{1}{t^{n+1}} \exp\left[z\left(t + \frac{1}{t}\right)\right] \, du, \tag{53}$$

where C is the contour of Fig. 3. Transforming C into a unit circle centered on the origin and writing $t = e^{i\theta}$ this simplifies to

$$I_n(z) = \frac{1}{\pi} \int_0^\pi \cos n\theta \; e^{z\cos\theta} \, d\theta. \tag{54}$$

IX. ASYMPTOTIC EXPANSIONS: THE LEADING ORDER TERM

One of the most important uses of the integral representation is to derive asymptotic expansions, valid for large $|z|$.

A. Expansions for $J_\nu, H_\nu^{(1)}$ and $H_\nu^{(2)}$.

It is convenient to start with the contour integral representation in Eq. (39) *i.e.*

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_c \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) \, dt. \tag{55}$$

Defining the function in the exponential to be $f(t)$, *i.e.*

$$f(t) = t - \frac{z^2}{4t}, \tag{56}$$

and defining θ to be the argument of z, *i.e.*

$$z = |z| e^{i\theta}, \tag{57}$$

then the saddle points of $f(t)$ are where $f'(t) = 0$ with

$$f'(t) = 1 + \frac{z^2}{4t^2}, \tag{58}$$

i.e.

$$t = t_\pm = \pm\frac{iz}{2} = \frac{|z|}{2} e^{i(\theta \pm \frac{\pi}{2})}. \tag{59}$$

The value of f and its second derivative, f'', at the two saddle points are given by

$$f(t_\pm) = \pm iz = |z| e^{i(\theta \pm \frac{\pi}{2})} \tag{60}$$

$$f''(t_\pm) = \mp\frac{4i}{z} = \frac{4}{|z|} e^{-i(\theta \pm \frac{\pi}{2})}. \tag{61}$$

Hence close to the saddle points we have

$$f(t_\pm + \delta t) = \pm iz + \frac{2}{|z|} e^{-i(\theta \pm \frac{\pi}{2})} \delta t^2 + O\left(\delta t^3\right). \tag{62}$$

We need the path of steepest descent, for which the change in f away from the saddle point is real and negative. Writing

$$\delta t = |\delta t| e^{i\phi_\pm}, \tag{63}$$

then ϕ_+ and ϕ_- are given by

$$\phi_\pm = \frac{\theta}{2} + \frac{\pi}{2} \pm \frac{\pi}{4}. \tag{64}$$

Note that $\phi_\pm$ is the angle that the path of steepest descent makes with the t_x axis. Hence, along the steepest descent path, Eq. (62) becomes

$$f(t_\pm + \delta t) = \pm iz - \frac{2}{|z|} \delta t^2 + O\left(\delta t^3\right). \tag{65}$$

For example, if z is real and positive, x say, then the saddle points are on the imaginary axis. Consider first the saddle point at $t = t_+ = ix/2$. Along the path of steepest descent the imaginary part of f stays constant and so, still assuming x is real and positive, we have, from Eq. (56),

$$t_y + \frac{x^2}{4} \frac{t_y}{t_x^2 + t_y^2} = x, \tag{66}$$

where $t = t_x + it_y$. Along these trajectories $t_y \to x$ as $|t| \to \infty$, and $t_y \to (4/x)t_x^2$ as $t \to 0$. The steepest descent trajectory, called C_+, is shown in fig. 3. The steepest descent trajectory for the saddle point at t_- is also shown and denoted by C_-.

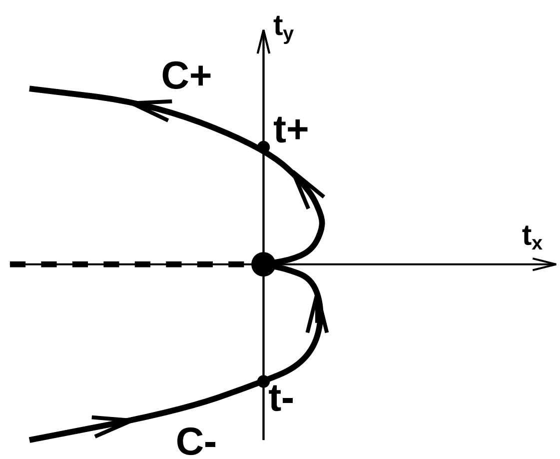

FIG. 4. The steepest descent trajectories in the complex t-plane, used in Eq. (55) to determine the asymptotic expansion of $J_\nu(z)$ for the case of $z(= x)$ real and positive. Here $t_+ = ix/2$ and $t_- = -ix/2$.

As $\arg z$ becomes non-zero the saddle points rotate in the complex t-plane according to Eq. (59) and the direction of the steepest descent paths also rotate according to Eq. (64).

For arbitrary z we write

$$\delta t \equiv t - t_+ = \frac{|z|}{2} r e^{i\phi_+} \tag{67}$$

with r real, so

$$dt = \frac{|z|}{2} e^{i\phi_+} dr. \tag{68}$$

Here we will just evaluate the leading asymptotic behavior, which means that we can replace the $t^{-(\nu+1)}$ prefactor by its value at the saddle point, *i.e.*

$$\frac{1}{t^{\nu+1}} \to \left(\frac{2}{z}\right)^{\nu+1} e^{-i(\nu+1)\frac{\pi}{2}}, \tag{69}$$

and use the expansion of $f(t)$ to second order about the saddle point given in Eq. (65). Collecting all the factors we get

$$\left(\frac{z}{2}\right)^{\nu} \frac{1}{2\pi i} \int_{C_+} \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) \, dt \approx \frac{1}{2\pi} e^{i(z - \frac{\theta}{2} - \frac{\nu\pi}{2} - \frac{\pi}{4})} \int_{-\infty}^{\infty} e^{-|z| r^2/2} dr \tag{70}$$

$$= \frac{1}{\sqrt{2\pi z}} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \; . \tag{71}$$

The contribution from the saddle point at t_- is evaluated along the same lines and simply gives Eq. (71) with i replaced by $-i$, *i.e.*

$$\left(\frac{z}{2}\right)^{\nu} \frac{1}{2\pi i} \int_{C_-} \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) \, dt \approx \frac{1}{\sqrt{2\pi z}} e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \; . \tag{72}$$

The total contribution is just the sum of Eq. (71) and (72), *i.e.*

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}) \left[1 + O\left(\frac{1}{z}\right)\right] \qquad (|\arg z| < \pi). \tag{73}$$

In §X we will write down the higher order terms, and we will discuss in in §XI the reason for the restriction on $\arg z$.

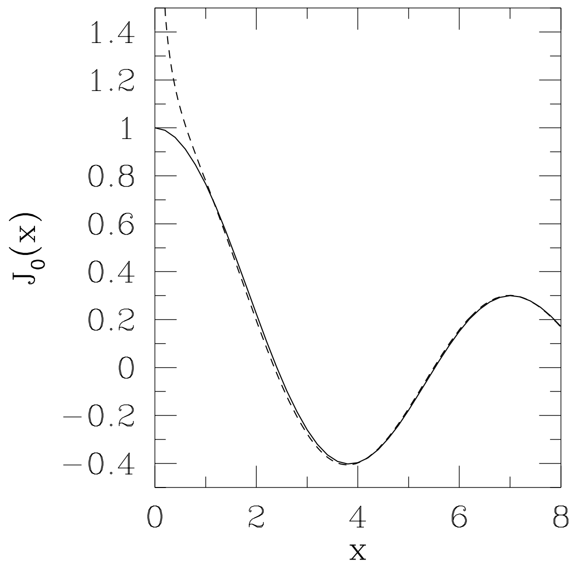

FIG. 5. The solid line is $J_0(x)$ and the dashed line is the leading term in the asymptotic expansion for $J_0(x)$, given by Eq. (73) with $\nu = 0$.

In Fig. 5 we compare the leading term in the asymptotic expansion for $J_0(x)$ obtained from Eq. (73), with the exact function $J_0(x)$, for x real and positive. One sees that the expansion converges quickly to $J_0(x)$ as x increases.

We can also determine asymptotic for the Hankel functions. Let us separate the contributions of the two steepest descent paths, C_+ and C_-,to $H_\nu^{(2)}(z)$, defined in Eq. (12). In an obvious notation

$$H_\nu^{(2)}(z) = H_\nu^{(2,+)}(z) + H_\nu^{(2,-)}(z), \tag{74}$$

where, from Eq. (12),

$$H_\nu^{(2,+)}(z) = \frac{i}{\sin \nu\pi} \left\{ J_{-\nu}^{+}(z) - e^{i\nu\pi} J_\nu^{+}(z) \right\}, \tag{75}$$

$$H_\nu^{(2,-)}(z) = \frac{i}{\sin \nu\pi} \left\{ J_{-\nu}^{-}(z) - e^{i\nu\pi} J_\nu^{-}(z) \right\}, \tag{76}$$

and from Eq. (55) we have

$$J_{-\nu}^{+}(z) = \left(\frac{z}{2}\right)^{-\nu} \frac{1}{2\pi i} \int_{C_+} \frac{1}{t^{-\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) dt \tag{77}$$

$$J_\nu^{+}(z) = \left(\frac{z}{2}\right)^{\nu} \frac{1}{2\pi i} \int_{C_+} \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) dt. \tag{78}$$

In the expression for $J_{-\nu}^{+}(z)$ we make the substitution $t \to e^{i\pi} z^2/(4t)$, which leaves $t - z^2/(4t)$ invariant, so

$$J_{-\nu}^{+}(z) = -e^{i\nu\pi} \left(\frac{z}{2}\right)^{\nu} \frac{1}{2\pi i} \int_{C_+'} \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) dt, \tag{79}$$

where C'_+ is the contour that C_+ transforms into under this substitution. The substitution maps the range $0 < \arg t < \pi$ into the same range (so we stay on the same branch of t^ν) and the steepest descent path C_+, Eq. (66), is unchanged under the substitution, except that the path is traversed in the opposite sense, so $\int_{C_+'} \cdots = -\int_C \cdots$. Consequently

$$J^+_{-\nu}(z) = e^{i\nu\pi} J^+_\nu(z), \tag{80}$$

so, from Eq. (75)

$$H_\nu^{(2,+)}(z) = 0. \tag{81}$$

Hence $H_\nu^{(2)}(z)$ is determined entirely from the steepest descent contour C_-.

Proceeding as above, but with the substitution $t \to e^{-i\pi} z^2/(4t)$, one also finds

$$J^-_{-\nu}(z) = e^{-i\nu\pi} J^-_\nu(z), \tag{82}$$

so, from Eqs. (74), (76), (81), (82) and (55)

$$H_\nu^{(2)} = \frac{1}{\pi i}\left(\frac{z}{2}\right)^\nu \int_{C_-} \frac{1}{t^{\nu+1}} \exp\left[t - \frac{z^2}{4t}\right] dt. \tag{83}$$

Eqs. (80), (82), (11) and (55) also show that

$$H_\nu^{(1)} = \frac{1}{\pi i}\left(\frac{z}{2}\right)^\nu \int_{C_+} \frac{1}{t^{\nu+1}} \exp\left[t - \frac{z^2}{4t}\right] dt \tag{84}$$

From Eq. (71), (72), (83) and (84) one has the asymptotic expressions (correct to leading order in $1/z$):

$$H_\nu^{(1)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)} \left[1 + O\left(\frac{1}{z}\right)\right] \qquad (-\pi < |\arg z| < 2\pi) \tag{85}$$

$$H_\nu^{(2)}(z) \approx \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)} \left[1 + O\left(\frac{1}{z}\right)\right] \qquad (-2\pi < |\arg z| < \pi). \tag{86}$$

See §XI below for a discussion of why there is a restriction on $\arg z$. We see that if $J_\nu(z)$ and $Y_\nu(z)$ are analogous to sines and cosines, then $H_\nu^{(1)}(z)$ is analogous to e^{iz} and $H_\nu^{(2)}(z)$ is analogous to e^{-iz}.

B. Expansions for I_ν and K_ν

Let us next discuss the asymptotic expansions for the modified Bessel function, $I_\nu(z)$ whose contour integral representation is given in Eq. (46), *i.e.*

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{1}{t^{\nu+1}} \exp\left(t + \frac{z^2}{4t}\right) dt, \tag{87}$$

Obviously, the saddle points are at

$$t_\pm = \pm\frac{z}{2}. \tag{88}$$

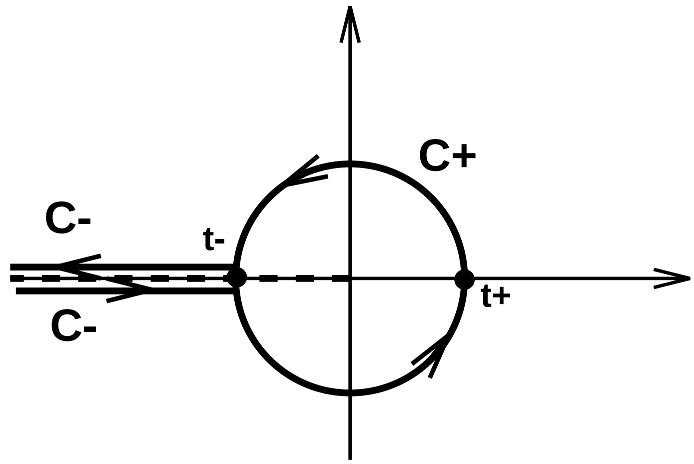

FIG. 6. The steepest descent trajectories in the complex t-plane used to determine the asymptotic expansion of $I_\nu(z)$ from Eq. (87) for the case of $z\,(=x)$ real and positive. the saddle points are at $t_+ = x/2$ and $t_- = -x/2$. The dashed line indicates the branch cut.

Let us assume, for now, that $z = x$ is real and positive so the saddle points are on the real axis. Then, repeating the saddle point analysis give above for $J_\nu(z)$, one finds that the steepest descent paths, along which $\mathrm{Im}\, t$ is constant, are

$$t_x^2 + t_y^2 = x^2, \qquad \text{for the saddle point at } t = t_+ = x/2 \tag{89}$$

$$t_y = 0, \qquad \text{for the saddle point at } t = t_- = -x/2. \tag{90}$$

A path which starts and ends with $t \to -\infty$ (on either side of the branch cut) and follows the steepest descent trajectories is shown in Fig. 6. The path C_+ is the circle and the path C_- is comprised of the two straight line segments one on each side of the negative real axis.

It is easy to see that

$$\exp\left(t + \frac{x^2}{4t}\right) = \exp(x) \qquad \text{at } t = t_+ = x/2 \tag{91}$$

$$\exp\left(t + \frac{x^2}{4t}\right) = \exp(-x) \qquad \text{at } t = t_- = -x/2, \tag{92}$$

so the path C_+ completely dominates the asymptotic expansion of $I_\nu(x)$. As in the calculation of the asymptotic expansion of $J_\nu(z)$ above, to leading order one replaces $t^{-(\nu+1)}$ by its saddle point and replaces the integral over $\exp\,(t + x^2/4t)$ by a Gaussian integral about the saddle point, which gives (replacing x by the complex variable z),

$$I_\nu(z) \approx \frac{1}{\sqrt{2\pi z}}\, e^z \left[1 + O\left(\frac{1}{z}\right)\right] \qquad (|\arg z| < \tfrac{\pi}{2}). \tag{93}$$

For a discussion of why there is a restriction on $\arg z$ see §XI below. Intuitively though, one can already say that the reason the expansion is only valid for $|\arg z| < \pi/2$ is that there is an exponentially small contribution to $I_\nu(z)$, from the saddle point at t_-, which is not included in the asymptotic expansion because e^{-z} is smaller than any inverse power of z if $|\arg z| < \pi/2$.

However, this piece *dominates* when $\frac{\pi}{2} < |\arg z| < \frac{3\pi}{2}$, so the asymptotic expansion clearly cannot be valid in this region.

At higher order in $1/z$ the terms *do* depend on ν, see §X, unlike the leading order term in Eq. (93). However, to all orders the expansions for $I_\nu(z)$ and $I_{-\nu}(z)$ are identical. The functions themselves are not identical, but, as we shall later in this section, the difference is exponentially small (for $\mathrm{Re}\, z > 0$) and so does not appear in an expansion in $1/z$.

Next we determine $K_\nu(x)$ from Eqs. (24) and Eq. (47). Following similar arguments to those used above to determine the contour integral representation of the Hankel functions, one can show that the contribution of C_+ to $I_\nu(x)$ cancels the contribution of C_+ to $I_{-\nu}(z)$ in Eq. (24). Hence $K_\nu(x)$ is entirely determined by C_-. Taking care to ensure that the functions are evaluated on the correct branch we find, from Eqs. (24) and (47)

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\frac{1}{2\pi i}\int_1^\infty \left\{(te^{i\pi})^\nu - (te^{-i\pi})^\nu - (te^{i\pi})^{-\nu} + (te^{-i\pi})^{-\nu}\right\} \exp\left[-\frac{x}{2}\left(t+\frac{1}{t}\right)\right]\frac{dt}{t} \tag{94}$$

$$= \tfrac{1}{2}\int_1^\infty \left(t^\nu + t^{-\nu}\right)\exp\left[-\frac{x}{2}\left(t+\frac{1}{t}\right)\right]\frac{dt}{t}. \tag{95}$$

With the substitution, $t \to e^t$ (and generalizing x to the complex variable z, which requires $|\arg z| < \pi/2$ for the integral to converge) this becomes

$$K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh\nu t\, dt, \qquad |\arg z| < \tfrac{\pi}{2}. \tag{96}$$

This is a useful representation of $K_\nu(z)$ for arbitrary ν. Analytic continuation to different values of $\arg z$ can be obtained by an appropriate rotation of the path of integration in the complex t-plane.

It is straightforward to get the leading term in the asymptotic expansion of $K_\nu(z)$ from Eq. (96). The saddle point is at $t = 0$ and, in the vicinity of $t = 0$, the integrand is $\exp(-z - zt^2/2)$. Hence, as usual, one has to do a Gaussian integral, but here one is restricted to just positive values of t, so we get only half the usual contribution. Since

$$\int_0^\infty \exp(-zt^2/2)\, dt = \tfrac{1}{2}\sqrt{\frac{2\pi}{z}}, \tag{97}$$

we obtain

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + O\left(\frac{1}{z}\right)\right] \qquad (|\arg z| < \tfrac{3\pi}{2}). \tag{98}$$

For a discussion on why there is a restriction on $\arg z$, see §XI. Note that $K_\nu(z)$, which is proportional to $I_\nu(z) - I_{-\nu}(z)$, is exponentially small (for $\mathrm{Re}\, z > 0$), so the expansions for $I_\nu(z)$ and $I_{-\nu}(z)$ are identical to all orders in $1/z$, as noted earlier in this section.

X. ASYMPTOTIC EXPANSIONS: HIGHER ORDER TERMS

In the previous section we derived the leading order asymptotic behavior of various Bessel functions for large $|z|$, by expanding the integrand about the saddle point(s) and performing a Gaussian integral. We shall now see how one can obtain higher orders in $1/z$.

It is simplest to consider first the expansion for $K_\nu(z)$, since it has a convenient real integral representation, Eq. (96). We start by writing Eq. (96) as a Gaussian integral by defining

$$\frac{u^2}{2} = \cosh t - 1 = 2\sinh^2\left(\frac{t}{2}\right), \tag{99}$$

so

$$t = 2\,\mathrm{arcsinh}\left(\frac{u}{2}\right). \tag{100}$$

Hence, Eq. (96) can be written

$$K_\nu(z) = e^{-z}\int_0^\infty e^{-zu^2/2}\cosh\left[\nu t(u)\right]\frac{dt}{du}\,du. \tag{101}$$

Since $\cosh(\nu t)\,dt/du = (1/\nu)d/du(\sinh \nu t)$, an integration by parts gives

$$K_\nu(z) = \frac{z}{\nu}e^{-z}\int_0^\infty ue^{-zu^2/2}\sinh\left[2\nu\,\mathrm{arcsinh}\,\frac{u}{2}\right]\,du. \tag{102}$$

Now we make the expansion

$$\sinh\left[2\nu\,\mathrm{arcsinh}\,\frac{u}{2}\right] = \sum_{n=0}^\infty a_n\left(\frac{u}{2}\right)^n \tag{103}$$

where the a_n are the coefficients in the expansion

$$\sinh(2\nu\theta) = \sum_{n=0}^\infty a_n \sinh^n\theta. \tag{104}$$

The a_n are determined by differentiating this last expression twice with respect to θ and equating the coefficients of $\sinh^n\theta$. This gives

$$4\nu^2 a_n = n^2 a_n + (n+1)(n+2)a_{n+2}, \tag{105}$$

or

$$a_{n+2} = \frac{(4\nu^2 - n^2)}{(n+1)(n+2)}a_n. \tag{106}$$

Clearly from Eq. (104) one has

$$a_0 = 0, \qquad a_1 = 2\nu, \tag{107}$$

so $a_{2n} = 0$ and

$$a_{2n+1} = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\cdots(4\nu^2 - (2n-1)^2)}{(2n+1)!}2\nu. \tag{108}$$

Substituting Eq. (103) into Eq. (102) one finds

$$K_\nu(z) \approx z e^{-z} \sum_{n=0}^{\infty} \frac{a_{2n+1}}{\nu} \int_0^\infty u e^{-zu^2/2} \left(\frac{u}{2}\right)^{2n+1} du \tag{109}$$

$$= e^{-z} \sum_{n=0}^{\infty} \frac{a_{2n+1}}{\nu} \sqrt{\frac{\pi}{2z}} \frac{1}{2^{2n+1}} \frac{(2n+1)!!}{z^n}, \tag{110}$$

where we used

$$\int_0^\infty u^{2n+2} e^{-u^2/2}\, dt = \sqrt{\frac{\pi}{2}}\, (2n+1)!! \tag{111}$$

Now

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!}, \tag{112}$$

so, on substituting Eqs. (108) and (112) into Eq. (110), we find

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + \frac{4\nu^2 - 1^2}{1!\, 8z} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2!\, (8z)^2} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)}{3!\, (8z)^3} + \cdots \right\}. \tag{113}$$

Thus we have been able to determine the asymptotic expansion of to *all* orders. To leading order, Eq. (113) agrees with Eq. (98) as required.

We can write Eq. (113) as

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} b_n \left(\frac{1}{z}\right)^n, \tag{114}$$

where, for large n, b_n is independent of ν and, using Stirling's approximation,

$$n! \approx n^n e^{-n} \sqrt{2\pi n} \left[1 + O\left(\frac{1}{n}\right)\right], \tag{115}$$

is given, for large n, by

$$b_n \approx \frac{1}{\pi} n^{-1} \left(-\frac{1}{2}\right)^n n! \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{116}$$

This is the typical form expected for an asymptotic series, with the n-th order term growing like $cn^b a^n n!$. Because of the $n!$, the series is only asymptotic and has zero radius of convergence. We find $a = -(1/2)$ to be negative, so successive terms alternate in sign and we anticipate that the series will be Borel summable. This is the case, because the steepest descent integration completely determines the function, (*i.e.* there are no other, exponentially small, saddle points).

It is not much more complicated to do a similar calculation for other Bessel functions, even though the integrand is complex. The reason is that the imaginary part of the function in the exponential is kept constant along a steepest descent path, so the integral can quite easily be transformed into a real integral similar to Eq. (102).

For example, the calculation for $J_\nu(z)$ gives

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \Bigg\{ \cos(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}) \left[1 - \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!\,(8z)^2} + \frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)(4\nu^2-7^2)}{4!\,(8z)^4} - \cdots \right]$$

$$- \sin(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}) \left[\frac{4\nu^2-1^2}{1!\,8z} - \frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{3!\,(8z)^3} + \cdots \right] \Bigg\}. \tag{117}$$

Note that the series in Eqs. (113) and (117) *terminate* if ν is a half-integer and the asymptotic expansions then give *exact* expressions for the spherical Bessel functions discussed in §VII.

XI. CHANGE IN ASYMPTOTIC FORM AS ARG Z CHANGES: STOKES' PHENOMENON

In §IX we stated that the asymptotic expansions we derived were only valid for a limited range of $\arg z$. Let is now try to understand this restriction.

Consider, for example, the leading term in the expansion for $J_\nu(z)$, shown in Eq. (73), *i.e.*

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}} \cos(z - \tfrac{\nu\pi}{2} - \tfrac{\pi}{4}) \qquad (|\arg z| < \pi) \tag{118}$$

$$= \sqrt{\frac{1}{2\pi z}} \left[e^{i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})} + e^{-i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})} \right]. \tag{119}$$

Let us take $z = x$, real and positive, where Eq. (119) is certainly valid, and then increase $\arg z$ by π. From Eq. (4) we see that $J_\nu(x)$ simply gets multiplied by $e^{i\nu\pi}$, so the correct asymptotic expansion is

$$J_\nu(e^{i\pi} x) \approx e^{i\nu\pi} \sqrt{\frac{1}{2\pi x}} \left[e^{i(x-\frac{\nu\pi}{2}-\frac{\pi}{4})} + e^{-i(x-\frac{\nu\pi}{2}-\frac{\pi}{4})} \right]. \tag{120}$$

However, if we take replace z by $xe^{i\pi}$ in our expansion, Eq. (119), we get a somewhat different result,

$$e^{i\nu\pi} \sqrt{\frac{1}{2\pi x}} \left[e^{i(x-\frac{\nu\pi}{2}-\frac{\pi}{4})} - e^{-2i\pi\nu} e^{-i(x-\frac{\nu\pi}{2}-\frac{\pi}{4})} \right]. \tag{121}$$

The first term is correct, but the second term has the wrong phase (unless ν is a half-integer). Something must go wrong with the asymptotic expansion in Eq. (119) at a value of $\arg z$ in the range $0 \le \arg z \le \pi$. This type of behavior was first discussed by Stokes, and so is generally called Stokes' phenomenon.

To study this we need to investigate the nature of the steepest descent path as $\arg z$ is changed. Let us start with the contour integral representation in Eq. (40), where for z real (actually $|\arg z| < \pi/2$) we can use the path in Fig. 2. For convenience, we make the substitution $t = e^u$, so

$$J_\nu(z) = \frac{1}{2\pi i} \int_C e^{z \sinh u - \nu u}\, du, \tag{122}$$

The path in the t-plane starts with $|t| \to \infty, \arg t = -\pi$ and ends with $|t| \to \infty, \arg t = \pi$, see Fig. 2, so the path in the u-plane must start with $u_x \to \infty, u_y = -\pi$ and must end with $u_x \to \infty, u_y = \pi$. A path which does this is shown by the dashed line in Fig. 7. For integer ν, the contributions from the horizontal sections cancel and the contribution from the vertical section is easily seen to reproduce Eq. (52).

The saddle points are where $\cosh u = 0$, *i.e.*

$$u = (n + \tfrac{1}{2})\pi i, \qquad n \text{ integer}. \tag{123}$$

For $\arg z = 0$ the path can be transformed into two steepest descent paths through the saddle points at $\pm \pi i/2$, as shown in Fig 7. To see this, note that for the saddle point at $u = -\pi i/2$, we need to find paths where $\mathrm{Im} \sinh u = -1$, *i.e.*

$$\cosh u_x \sin u_y = -1. \tag{124}$$

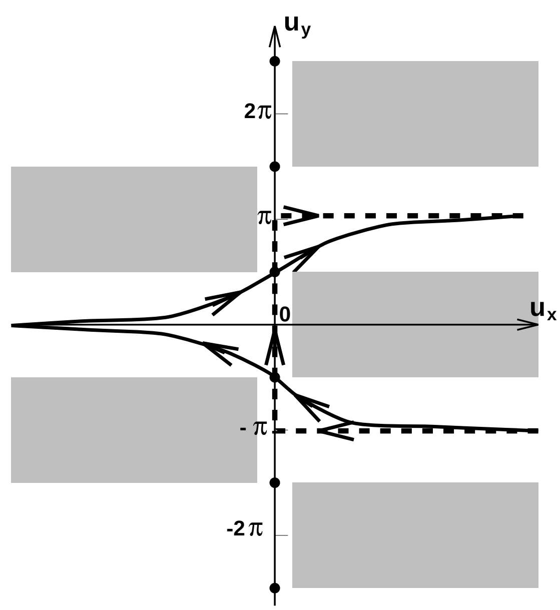

FIG. 7. The dashed line is a path in the complex u-plane which can be used to determine $J_\nu(z)$ for large $|z|$ with $\arg z = 0$, according to Eq. (122). The saddle points, which are at $u = (n + 1/2)\pi i$ for integer n are indicated by the circles. The solid line shows the path deformed so that it follows steepest descent paths through the saddle points at $u = \pm i\pi/2$ (which correspond to the saddle points t_+ and t_- in the complex t plane in Fig. 4). The connection between t in Fig. 4 and u is $t = (z/2)e^u$. The shaded regions indicate where the absolute value of the integrand gets large and diverges as $|u_x| \to \infty$. These regions must therefore be avoided by the path at large $|u_x|$ since the integrand must vanish at infinity. In the region between the shaded areas, the integrand in Eq. (122) tends to zero exponentially fast as $|u_x| \to 0$.

As $u_x \to \pm\infty$ one has $\sin u_y \to 0$ and it is not hard to show, and is obvious from location of the shaded regions of Fig. 7 (where $\mathrm{Re}(z \sinh u)$ is large and positive) that, on the steepest descent path, (as opposed to the steepest ascent path) $u_y \to -\pi$ as $u_x \to \infty$ and $u_y \to 0$ as $u_x \to -\infty$. Furthermore it is easy to see that the path makes an angle of $3\pi/4$ with the positive real axis. Similar considerations apply to the steepest descent path through $u = \pi i/2$. Hence the original path can be deformed so that it is comprised *entirely* of the two steepest descent paths. The combination of the two steepest descent paths through the saddle points gives the asymptotic expansion in Eq. (119).

Clearly the path in Fig. 7 could start with any u_y in the range

$$-\frac{3\pi}{2} < u_y < -\frac{\pi}{2} \tag{125}$$

and could end with

$$\frac{\pi}{2} < u_y < \frac{3\pi}{2}, \tag{126}$$

since, in these regions, $z \sinh u \to -\infty$ as $|u_x| \to \infty$, so the integrand in Eq. (122) tends to zero in this limit, see Fig. 7. If we now make $\arg z$ non-zero, then we have to ensure that the path stays in the region where $z \sinh u \to -\infty$ as $|u_x| \to \infty$ and lies within the range specified in Eqs. (125) and (126) for $\arg z \to 0$.

This means that the path must start with u_y in the range

$$-\frac{3\pi}{2} + \arg z < u_y < -\frac{\pi}{2} + \arg z \tag{127}$$

and must end with

$$\frac{\pi}{2} - \arg z < u_y < \frac{3\pi}{2} - \arg z. \tag{128}$$

Let us now consider how the steepest descent trajectory through $u = -\pi i/2$ changes if we increase $\arg z$ from zero. Defining

$$f(u) = z \sinh u \tag{129}$$

and

$$z = |z| e^{i\theta}, \qquad \theta = \arg z \tag{130}$$

then, for small $\delta u \equiv u - (-i\pi/2) = |\delta u| e^{i\phi}$, we have

$$f(u) = |z| e^{i\theta - \frac{\pi}{2}} + \frac{|z|}{2} e^{i(\theta - \frac{\pi}{2} + 2\phi)} |\delta u|^2 + O\left(\delta u^3\right). \tag{131}$$

Along the path of steepest descent, the change in f must be real and negative, so the angle the path makes with the positive real axis is given by

$$\phi = \frac{3\pi}{4} - \frac{\arg z}{2}. \tag{132}$$

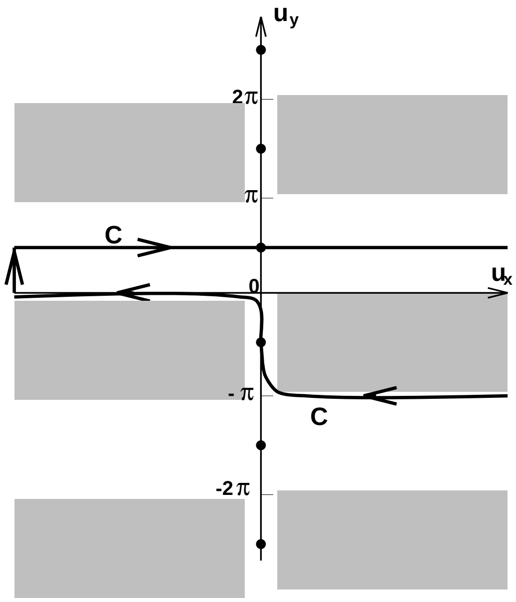

FIG. 8. The path in the complex u-plane for the evaluation of $J_\nu(z)$ from Eq. (122) for the case of $\arg z = \pi/2 - \delta$, (δ small and positive). The path is made up of two steepest descent paths through the saddle points at $u = \pm \pi i/2$. In the short vertical section at the left of the figure $u_x \to -\infty$, so this gives a vanishing contribution.

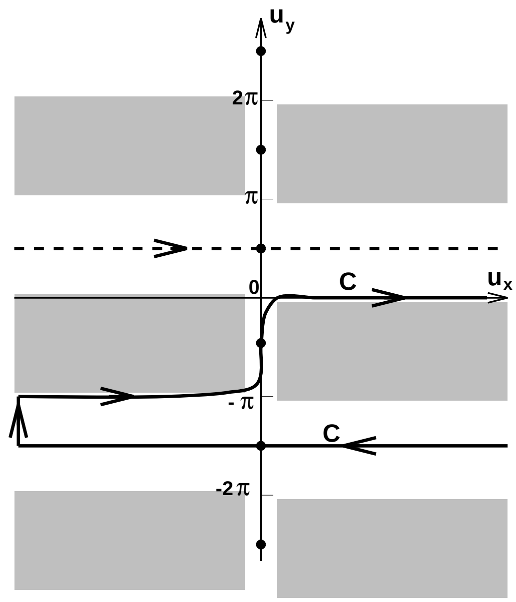

FIG. 9. The path in the complex u-plane for the evaluation of $J_\nu(z)$ from Eq. (122) for the case of $\arg z = \pi/2 + \delta$, (δ small and positive). The path can no longer be deformed so that it is made up of steepest descent paths through the saddle points at $u = \pm \pi i/2$. The trajectory still follows the steepest descent path through the saddle point at $u = -\pi i/2$ but it no longer follows the steepest descent path through the saddle point at $u = \pi i/2$, shown by a dashed line. This has been replaced by a steepest descent path through the saddle point at $u = -3\pi i/2$. As in Fig. 8, the short vertical section at the left of the figure gives a vanishing contribution.

As $\arg z \to \pi/2$ from below, the direction of the steepest descent path at $u = -\pi i/2$ approaches the imaginary axis. This is shown in Fig. 8. As long as $0 < \arg z < \pi/2$ the whole trajectory can still be deformed into steepest descent paths at $u = \pm\pi i/2$, see Fig. 8. Thus the saddle point integrals (evaluated to infinite order in $1/z$, see Eq. (117)) *completely* determine the function. Consequently the analytic continuation of the asymptotic series (valid for $\arg z = 0$) will still be correct. The series will be Borel summable in this region *i.e.* the Borel sum will give the exact answer. (Borel summation is a way of determining the sum of an asymptotic series.)

However, if $\arg z$ exceeds $\pi/2$ the steepest descent path through $u = -\pi i/2$ crosses the imaginary axis in the opposite sense, as shown in Fig. 9. This means that the overall structure of the trajectory changes when $\arg z$ passes $\pi/2$ as can be seen by comparing Figs. 8 and 9. The trajectory is no longer comprised of steepest descent paths at $u = \pm\pi i/2$. There is still a steepest descent path going through the saddle point at $u = -\pi i/2$, which gives the correct analytic continuation of the result from that saddle point obtained for $\arg z = 0$. However, the saddle point at $\pi i/2$ has been replaced by one at $-3\pi i/2$. This does *not* give the analytic continuation of the contribution from the $\pi i/2$ saddle point. Nonetheless, there is actually no change in the asymptotic expansion at $\arg z = \pi/2$ because the saddle point at $u = -\pi i/2$ gives a contribution of order e^y (with $z = x + iy$) which dominates the contribution from the saddles at $\pi i/2$ and $-3\pi i/2$, which are of order e^{-y}. Thus, to all orders in $1/z$, the asymptotic expansion in Eq. (119) is correct for $0 \le \arg z < \pi$. However, for $\pi/2 < \arg z < \pi$, there are *exponentially* small corrections which are *not* correctly given by the analytic continuation of the expansion. Thus the series is not Borel summable in this region.

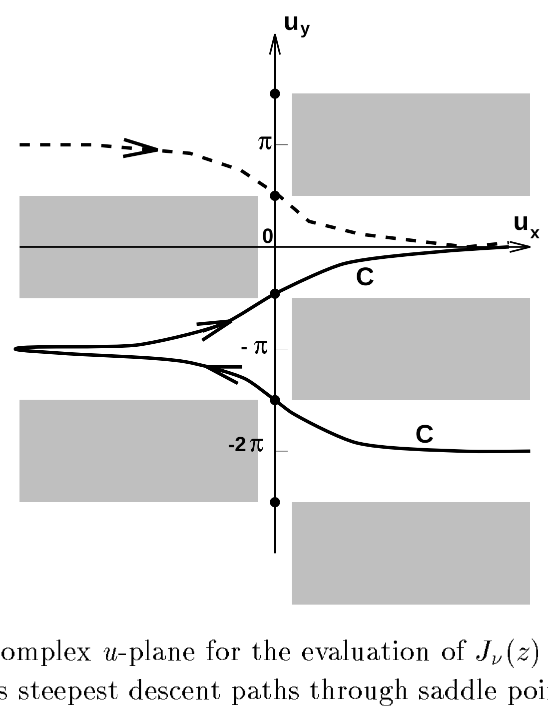

FIG. 10. The path in the complex u-plane for the evaluation of $J_\nu(z)$ from Eq. (122) for the case of $\arg z = \pi$, The path comprises steepest descent paths through saddle points at $u = -\pi i/2$ and $-3\pi i/2$. Evaluating the contributions from these steepest descent paths gives the correct asymptotic expansion for $J_\nu(z)$ in Eq. (120), where z is written $e^{i\pi}x$. This is not the analytic continuation of the expansion obtained for $\arg z = 0$, which is Eq. (121), because the latter is the analytic continuation of steepest descent paths through $u = \pm\pi i/2$. The steepest descent path through $\pi i/2$, shown by the dashed line, is not on the trajectory used to calculate $J_\nu(z)$.

For $\arg z = \pi$, see Fig. 10, the contribution from the saddle point at $u = \pi i/2$ is no longer exponentially small compared with the contribution from the saddle point at $u = -\pi i/2$, but rather the two are equal in magnitude. Hence the analytic continuation of the expansion obtained for $\arg z = 0$, is *not* valid for $\arg z = \pi$, as we saw explicitly above by comparing Eq. (120) with Eq. (121). One can verify that the steepest descent path through the saddle point at $-3\pi i/2$ does give the second term in Eq. (120). The reason for the extra phase factor $-\exp(-2\pi i\nu)$ in the offending term in Eq. (121) is that the steepest descent path through $\pi i/2$ goes in the opposite sense to that through $-3\pi i/2$, see Fig. 10, (which explains the minus sign) and is shifted up the u_y axis by $2\pi i$ and so, according to Eq. (122) picks up a phase factor $\exp(-2\pi i\nu)$.

Similar considerations apply for $\arg z < 0$ and indicate that the asymptotic expansion is valid for $-\pi < \arg z \leq 0$, but that there is an exponentially small contribution, not included in the expansion, for $-\pi < \arg z < -\pi/2$. The series is therefore not Borel summable in this latter region.

To summarize, the asymptotic expansion for $J_\nu(z)$ in Eq (119) is valid for $|\arg z| < \pi$, as stated without proof in Eqs. (73) and (119), but the series is only Borel summable for $|\arg z| < \pi/2$. Similar considerations apply to the other asymptotic expansions that we have derived, and give rise to the bounds on $\arg z$ stated in Eqs. (85), (86), (93) and (98).

XII. REFERENCES

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