Homework 7, Solutions

- **1 (a)**

```
sphericalBesselj[n_, x_] := Sqrt[Pi / 2] / Sqrt[x] BesselJ[n + 1 / 2, x] // Apart

sphericalBessely[n_, x_] := Sqrt[Pi / 2] / Sqrt[x] BesselY[n + 1 / 2, x] // Apart
```

Note: I wrote the factor involving the square root of x as $1/\sqrt{x}$ rather than $\sqrt{1/x}$ to avoid having square roots that *Mathematica* won't cancel. (Alternatively I could have used **Simplify** with the condition that x > 0.) **Apart** presents the result in a neater way.

- **1 (b)**

```
{sphericalBesselj[0, x], sphericalBessely[0, x]}
```

$$\left\{\frac{\text{Sin}[x]}{x}, -\frac{\text{Cos}[x]}{x}\right\}$$

```
{sphericalBesselj[1, x], sphericalBessely[1, x]}
```

$$\left\{-\frac{\text{Cos}[x]}{x} + \frac{\text{Sin}[x]}{x^2}, -\frac{\text{Cos}[x]}{x^2} - \frac{\text{Sin}[x]}{x}\right\}$$

```
sphericalBesselj[10, x]
```

$$-\frac{55\left(11\,904\,165 - 1\,670\,760\,x^2 + 51\,597\,x^4 - 468\,x^6 + x^8\right)\text{Cos}[x]}{x^{10}} - \frac{\left(-654\,729\,075 + 310\,134\,825\,x^2 - 18\,918\,900\,x^4 + 315\,315\,x^6 - 1485\,x^8 + x^{10}\right)\text{Sin}[x]}{x^{11}}$$

```
sphericalBessely[10, x]
```

$$\frac{\left(-654\,729\,075 + 310\,134\,825\,x^2 - 18\,918\,900\,x^4 + 315\,315\,x^6 - 1485\,x^8 + x^{10}\right)\text{Cos}[x]}{x^{11}} - \frac{55\left(11\,904\,165 - 1\,670\,760\,x^2 + 51\,597\,x^4 - 468\,x^6 + x^8\right)\text{Sin}[x]}{x^{10}}$$

- **1 (c)**

```
Series[sphericalBesselj[10, x], {x, 0, 15}]
```

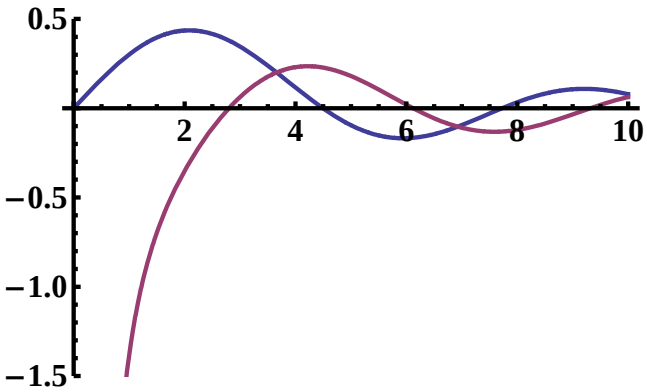
$$\frac{x^{10}}{13\,749\,310\,575} - \frac{x^{12}}{632\,468\,286\,450} + \frac{x^{14}}{63\,246\,828\,645\,000} + O[x]^{16}$$

```
Series[sphericalBessely[10, x], {x, 0, -6}]
```

$$-\frac{654\,729\,075}{x^{11}} - \frac{34\,459\,425}{2\,x^9} - \frac{2\,027\,025}{8\,x^7} + \frac{1}{O[x]^5}$$

1 (d)

```
Plot[{sphericalBesselj[1, x], sphericalBessely[1, x]}, {x, 0, 10}, PlotRange → {-1.5, 0.5}]
```



2 (a)

```
ser = Series[Tanh[x], {x, 0, 20}]
```

$$x - \frac{x^3}{3} + \frac{2\,x^5}{15} - \frac{17\,x^7}{315} + \frac{62\,x^9}{2835} - \frac{1382\,x^{11}}{155\,925} + \frac{21\,844\,x^{13}}{6\,081\,075} - \frac{929\,569\,x^{15}}{638\,512\,875} + \frac{6\,404\,582\,x^{17}}{10\,854\,718\,875} - \frac{443\,861\,162\,x^{19}}{1\,856\,156\,927\,625} + O[x]^{21}$$

2 (b)

```
InverseSeries[ser]
```

$$x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \frac{x^{17}}{17} + \frac{x^{19}}{19} + O[x]^{21}$$

Note: $\tanh^{-1}(x)$ = (1/2)ln[(1+x)/(1-x)], and we indeed recover the series for this function.

3 (a)

At low T the upper limit of the integral can be replaced by ∞. The integral is

```
Integrate[ x^4 Exp[x] / (Exp[x] - 1)^2, {x, 0, Infinity}]
```

$$\frac{4\,\pi^4}{15}$$

and multiplying by 9 gives the answer.

```
N[12 Pi^4 / 5.]
```

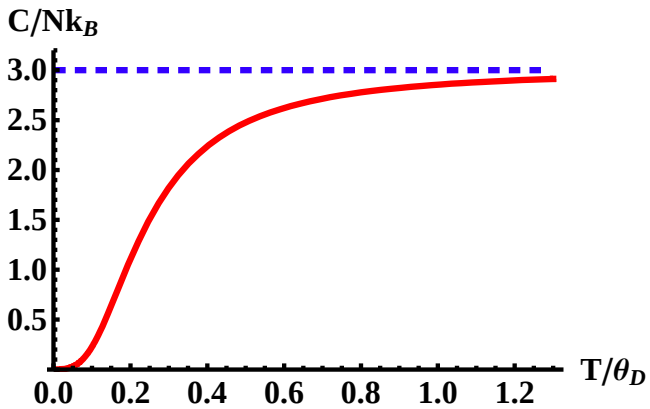
```
233.782
```

- **3 (b)**

At high-T the upper limit of the integral is small so we can expand the integrand, which becomes x^2. The integral is then trivial and leads to the answer given in the question.

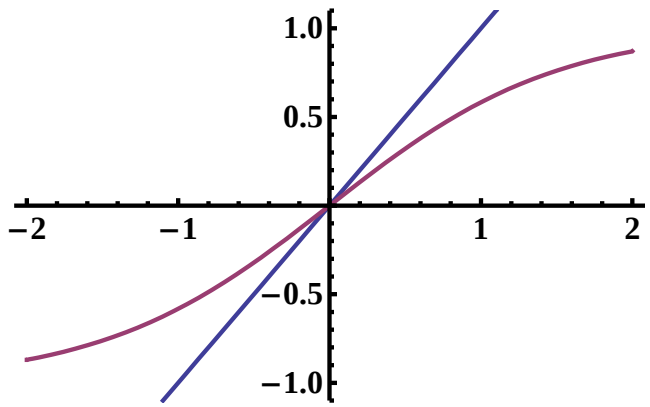
- **3 (c)**

```
Plot[{9 t^3 NIntegrate[x^4 e^x / (e^x - 1)^2, {x, 0, 1/t}], 3 + 0 t}, {t, 0.01, 1.3},
 AxesLabel → {"T/θD", "C/NkB"}, PlotRange → {0, 3.2}, PlotStyle →
  {{Hue[0], AbsoluteThickness[3]}, {Hue[0.7], AbsoluteThickness[3], Dashing[{0.01, 0.03}]}}]
```



- **4(a)**

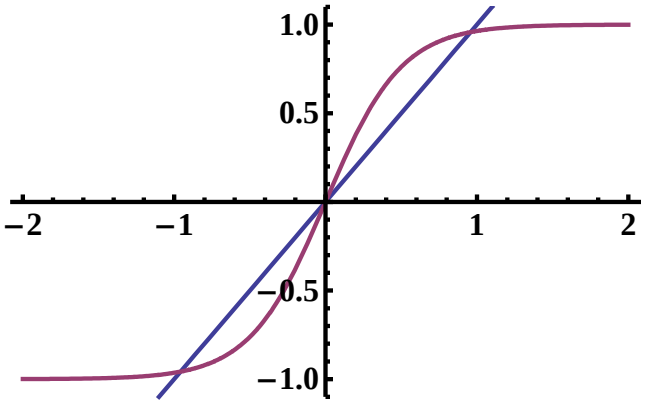
```
Plot[{m, Tanh[m/T] /. T → 1.5}, {m, -2, 2}, PlotRange → {-1.1, 1.1}]
```



The only solution is for m = 0. This will be the case for all T > 1 because tanh(x) = x for small x.

- **4(b)**

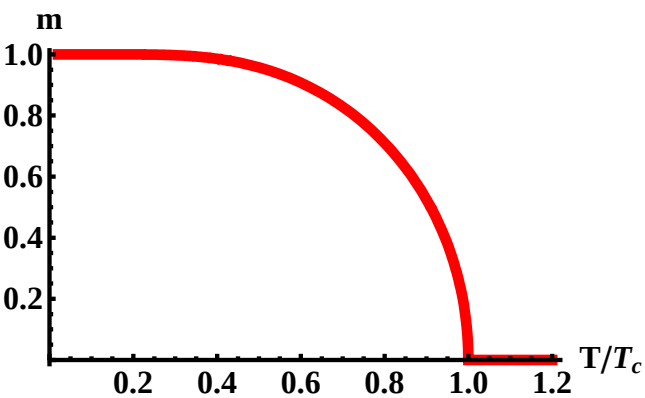
```
Plot[{m, Tanh[m/T]} /. T → 0.5, {m, -2, 2}, PlotRange → {-1.1, 1.1}]
```



We see that there are now two additional solutions with equal and opposite magnetization.

- **4(c)**

```
Plot[m /. FindRoot[m == Tanh[m/T], {m, 0.5}], {T, 0.02, 1.2},
 AxesLabel → {"T/Tc", "m"}, PlotStyle → {Hue[0], AbsoluteThickness[5]}]
```



- **4(d)**

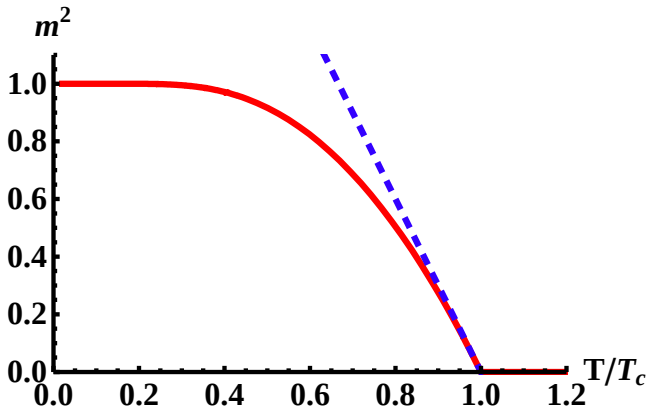
Since tanh(x) = x - $x^3/3$ + ... , we have

$$m\left(1 - \frac{J}{T}\right) = \frac{1}{3}\left(\frac{J}{T}\right)^3 m^3$$

for m small. Either m = 0 or $m^2 = 3\,(T_c - T)/T_c$ (to lowest order in $T_c - T$) with T_c = J.

4(e)

```
Plot[{m^2 /. FindRoot[m == Tanh[m/T], {m, 0.5}], 3 (1 - T)}, {T, 0.02, 1.2},
 AxesLabel → {"T/T_c", "m^2"}, PlotRange → {{0, 1.2}, {0, 1.1}}, PlotStyle →
  {{Hue[0], AbsoluteThickness[3]}, {Hue[0.7], AbsoluteThickness[3], Dashing[{0.01, 0.03}]}}]
```



The dashed line is the expected result for T close to T_c. It agrees well with the full numerical results shown in the solid line.

5(a)

This solution follows closely the handout on "The range of a baseball including air resistance" which should be consulted for further details.

```
Clear["Global`*"]
```

```
g = 9.81;
```

We determine the time, t_{final}, when the particle hits the ground as a function of θ .

```
tfinal[theta_] := (sol = NDSolve[
    {x''[t] == -k x'[t] Sqrt[y'[t]^2 + x'[t]^2], y''[t] == -k y'[t] Sqrt[y'[t]^2 + x'[t]^2] - g,
     x[0] == 0, y[0] == 0, x'[0] == v Cos[theta], y'[0] == v Sin[theta]}, {x, y}, {t, 0, 20}];
  yy[t_] = y[t] /. sol[[1]]; xx[t_] = x[t] /. sol[[1]];
  t /. FindRoot[yy[t], {t, 0.1, 20}, MaxIterations → 50])
```

Next we determine the horizontal distance traveled , x_{final},when the particle hits the ground (i.e. the range) as a function of θ .

```
xfinal[theta_?NumericQ] := xx[tfinal[theta]]
```

We test this function for v = 20, k = 0.

```
v = 20; k = 0; Table[xfinal[θ], {θ, 0.1, 1.0, 0.1}]
```

```
{8.10069, 15.8784, 23.0231, 29.25, 34.3107, 38.0036, 40.1814, 40.7573, 39.7084, 37.0763}
```

by checking that it gives $v^2 \sin 2\theta$ / g,

```
Table[v^2 Sin[2 θ] / g, {θ, 0.1, 1.0, 0.1}]
```

```
{8.10069, 15.8784, 23.0231, 29.25, 34.3107, 38.0036, 40.1814, 40.7573, 39.7084, 37.0763}
```

and indeed it does. We next optimize the range with respect to θ . The maximum range, x_{max} will be evaluated as a function of v. We will also compute the optimal angle θ_{max}.Note how we *indirectly* pass the value of v in the argument of **xmax** and **thetamax** to the function **xfinal[theta]**.

```
xmax[vv_] := (v = vv; -FindMinimum[-xfinal[theta], {theta, 0.1, 1.3}][[1]])
```

```
thetamax[vv_] := (v = vv; theta /. FindMinimum[-xfinal[theta], {theta, 0.1, 1.3}][[2]])
```

We verify that with no air resistance,k=0,we correctly obtain $x_{max}= v^2/ g$, $\theta_{max}= \pi /4$:

```
xmax[10]
```

```
10.1937
```

```
thetamax[10]
```

```
0.785398
```

From now on, we set k = 0.004, as stated in the question.

```
k = 0.004;
```

First we consider xmax for v = 50, with and without friction.

```
v = 50;
```

```
{xmax[v], v^2 / g}
```

```
{150.21, 254.842}
```

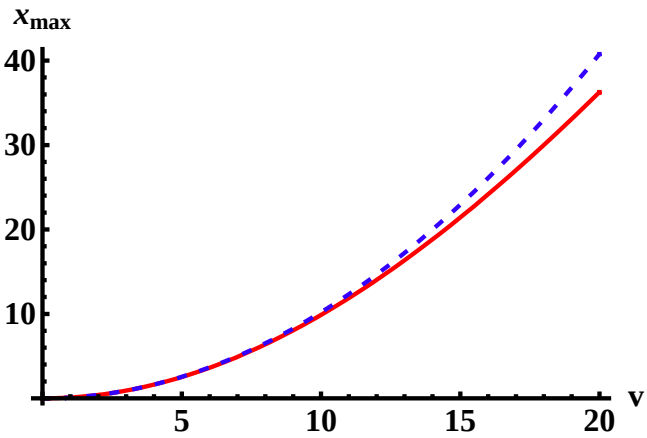
Hence without friction the maximum range for initial speed v = 50 m/sec would be 255 meters while with a friction coefficient of k=0.004 it is reduced to 150 meters.

- **5(b)**

Next we plot the results. The dashed line in the plots gives the result for no friction (i.e.k=0), $x_{max}= v^2/ g$.

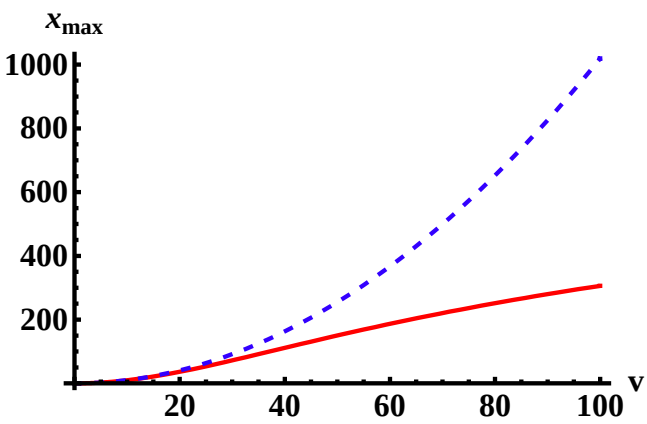
First of all we show that the effects of friction are small for small v.

```
Plot[{xmax[v], v^2/g}, {v, 0, 20}, AxesLabel → {"v", "x_max"}, PlotStyle →
  {{Hue[0], AbsoluteThickness[2]}, {Hue[0.7], AbsoluteThickness[2], Dashing[{0.01, 0.03}]}}]
```



Next we show that friction reduces the range considerably at large v.

```
Plot[{xmax[v], v^2/g}, {v, 0, 100}, AxesLabel → {"v", "x_max"}, PlotStyle →
  {{Hue[0], AbsoluteThickness[2]}, {Hue[0.7], AbsoluteThickness[2], Dashing[{0.01, 0.03}]}}]
```



- **6(a)**

In[1]:=
```
f[x_] := 4 λ x (1 - x)
```

(i) $\lambda = 0.2$

```
λ = 0.2; FixedPoint[f, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &) ]
```

3.49271×10^{-11}

i.e fixed point (FP) at zero x for $\lambda = 0.2$.

(ii) $\lambda = 0.6$

```
λ = 0.6; FixedPoint[f, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &) ]
```

```
0.583333
```

i.e fixed point (FP) at non-zero x for λ = 0.6.

```
f2[x_] := f[ f[x] ]; f4[x_] := f2[ f2[x] ]
```

(iii) λ = 0.8
Either plot successive results for x, which clearly shows a cycle of length 2, or show that FixedPoint for f does not converge, but **FixedPoint** for the second iterate, f2(x), does coverge.

```
λ = 0.8; FixedPoint[f2, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &) ]
```

```
0.799455
```

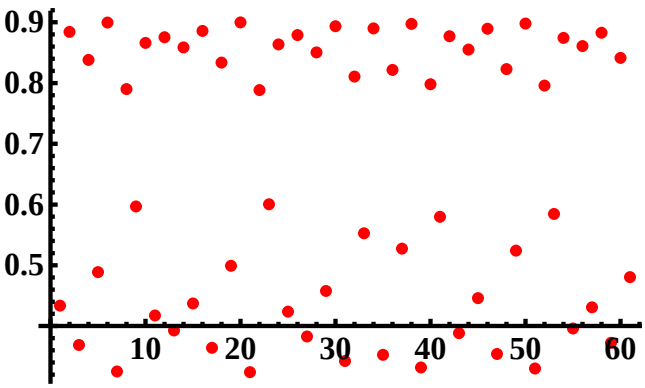
(iv) λ = 0.84
Either plot successive results for x, which clearly shows a cycle of length 2, or show that FixedPoint for f does not converge, but **FixedPoint** for f2(x), does coverge.

```
λ = 0.84; FixedPoint[f2, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &) ]
```

```
0.462375
```

(v) λ = 0.90
FixedPoint does not converge for f, f2, f4 or any other iterate of f that one can try. The data looks chaotic

```
λ = 0.9'; ListPlot[Drop[NestList[f, 0.7, 80], 20], PlotStyle → {Hue[0], PointSize[0.02]}]
```



so we conclude that there is chaos for λ = 0.90

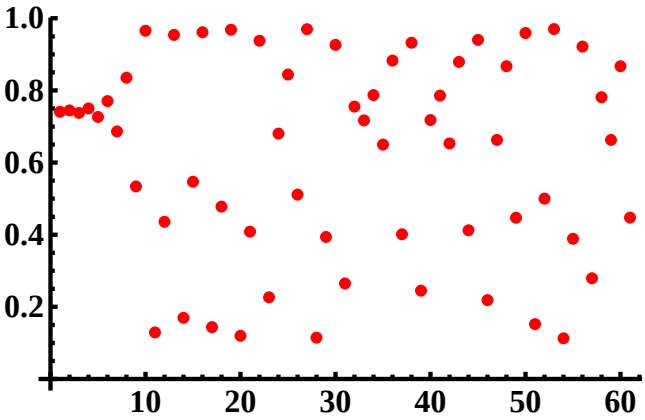
(vi) λ = 0.961
Looking at the plot in the handout, this looks like a period-3 cycle. If we try to verify this using **FixedPoint** on the third iterate, f(x), we find that it does not converge. Actually 0.961 is just in the region where the period three has doubled to period 6, though this is difficult to see from the plots. One finds that **FixedPoint** with with f6(x) does converge:

```
λ = 0.961; f3[x_] := f[f[f[x]]]; f6[x_] := f3[f3[x]];
FixedPoint[f6, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &) ]
```

```
0.498668
```

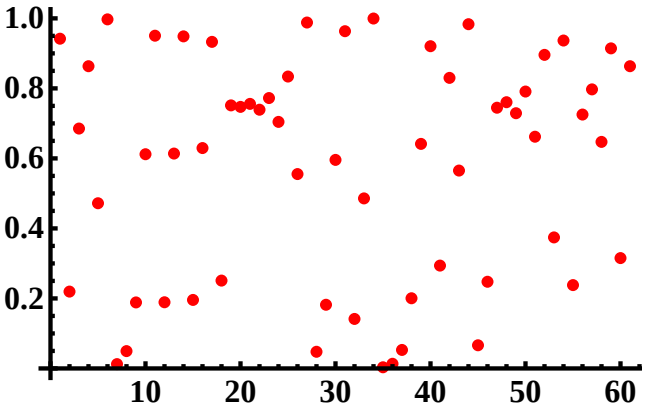
(vii) λ = 0.97
Chaos. Show a **ListPlot**

```
λ = 0.97; ListPlot[Drop[NestList[f, 0.7, 80], 20], PlotStyle → {Hue[0], PointSize[0.02]}]
```



(viii) $\lambda = 1$
Chaos. Show a **ListPlot**

```
λ = 1; ListPlot[Drop[NestList[f, 0.7, 80], 20], PlotStyle → {Hue[0], PointSize[0.02]}]
```



- **6(b)**

Here we compute the Lyapunov exponent, λ_L, as discussed in class. First of all we define a function **lya[...]** which will determine λ_L for a given value of λ . Note the use of the *Mathematica* construct **Apply [Plus, list]** to sum the elements of a list, and also the function **Length[list]**, which gives the number of elements in the list.

```
fp[x_] := f'[x]
```

```
lya[la_, xinit_, n_, ndrop_] := (λ = la; xlist = Drop[ NestList[f, xinit, n] , ndrop + 1 ];
  Apply[ Plus, Log[ Abs[ fp[xlist] ] ] ] / Length[xlist])
```

Next we set up a list of the λ values, and the corresponding Lyapunov exponents:

```
lvals = {0.2, 0.6, 0.8, 0.84, 0.9, 0.961, 0.97, 1}
```

```
{0.2, 0.6, 0.8, 0.84, 0.9, 0.961, 0.97, 1}
```

```
lyapunovs = Table[ lya[lvals[[n]], 0.7, 20 000, 1000] , {n, Length[lvals]} ]
```

```
{-0.223144, -0.916291, -0.916291, -0.28141, 0.181912, -0.280362, 0.46721, 0.693172}
```

Finally we display a pretty table of the results.

```
Print[" λ          λL   "]; Print[TableForm[Transpose[{lvals, lyapunovs}]]]
```

```
 λ          λL

0.2    -0.223144
0.6    -0.916291
0.8    -0.916291
0.84   -0.28141
0.9    0.181912
0.961  -0.280362
0.97   0.46721
1      0.693172
```

Note that λ_L is negative where the trajectory is a fixed point or limit cycle, i.e. λ = 0 (FP at x=0), λ = 0.6 (FP at non-zero x), λ = 0.8 (period-2 cycle), λ = 0.84 (period 2 cycle), and λ = 0.961 (period 6 cycle). However λ_L is positive in the chaotic regime, i.e. for λ = 0.9, 0.97 and 1.

For a FP at x =0, λ_L should equal log(4 λ),(show this by evaluating the derivative at the fixed point) and the result for λ = 0.2 agrees with this. For a FP at non-zero x, $\lambda_L$should equal log(| 2 - 4 λ |) (again show this by evaluating the derivative at the FP), and the result for λ = 0.6 agrees with this. For λ = 1, the Lyapunov exponent appears, numerically, to equal log(2). This result can also be shown analytically.

■ 6(c)

By inspection of the figure in the handout, we try the region round 0.935.

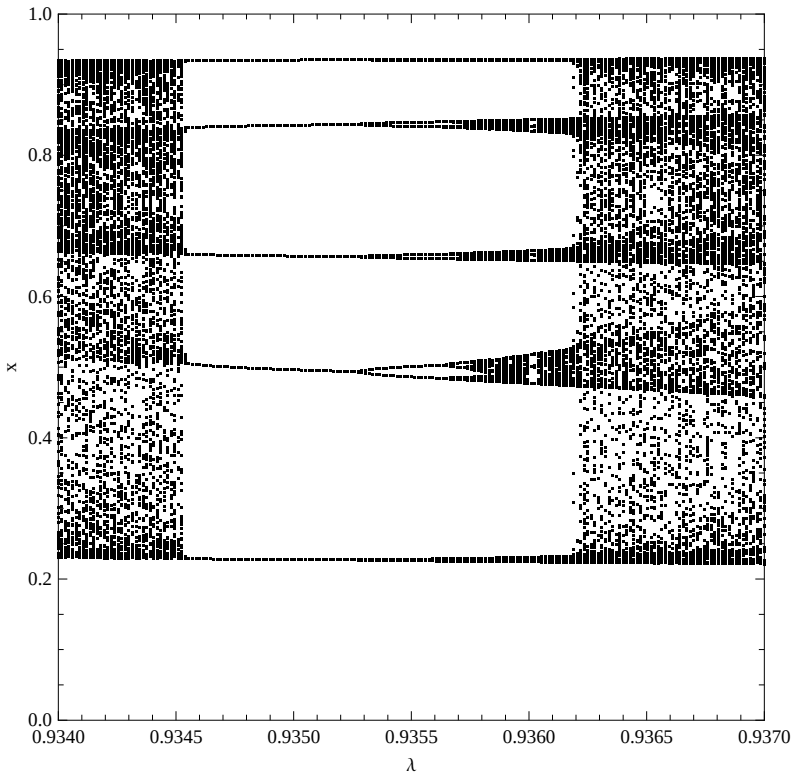
In[2]:=
```
iterate[m_, n_] := Drop[NestList[f, 0.5, n], m]
```

In[3]:=
```
drawpt[y_] := Point[{λ, y}]
```

In[4]:=
```
graph[λmin_, λmax_, nλ_, mdrop_, n_] := Graphics[{PointSize[0.001],
   Table[Map[drawpt, iterate[mdrop, n]], {λ, λmin, λmax, (λmax - λmin) / nλ}]}]
```

In[6]:=
```
Show[graph[0.934, 0.937, 200, 300, 500], Axes → False, Frame → True,
 FrameLabel → {"λ", "x"}, PlotRange → {{0.934, 0.937}, {0, 1}}, AspectRatio → 1]
```

Out[6]=

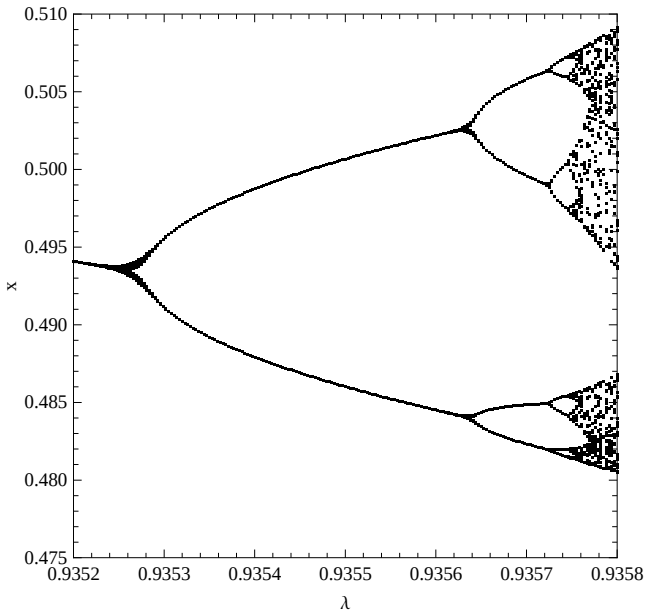


We see that this does indeed contain a period-5 cycle.

- **6(d)**

Now we blow up the cycle.

```
Show[graph[0.9352, 0.9358, 200, 300, 600], Axes → False, Frame → True,
 FrameLabel → {"λ", "x"}, PlotRange → {{0.9352, 0.9358}, {0.475, 0.51}}, AspectRatio → 1]
```



We see period doubling.

■ 6(e)

Define λ_k to be the value of λ where the 2^{k-1}cycle becomes unstable and goes to into a period 2^kcycle. We have λ_0= 1/4, and λ_1= 3/4. By using FixedPoint on f2 and f4 we can determine, with fair precision when the length-2 cycle becomes unstable. Using bisection (done by hand!) I deduce that **FixedPoint** for f2 converges for λ = 0.8623

```
λ = 0.8623; FixedPoint[f2, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &)]
```

```
0.440028
```

but not for λ = 0.8624, though f4 does have a fixed point for λ = 0.8624;

```
λ = 0.8624; FixedPoint[f4, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &)]
```

```
0.442748
```

Hence I estimate λ_2 = 0.86235. Kinzel and Reents give the exact result as $\left(1+\sqrt{6}\right)/4$ = 0.862372... . This is also shown using *Mathematica* in the handout on the logistic map.

Next look for the FP of f4(x) going instable. f4 has a fixed point for λ = 0.886

```
λ = 0.886; FixedPoint[f4, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &)]
```

```
0.363324
```

but not for λ = 0.8861, though f8 does have a fixed point for λ = 0.8861;

```
f8[x_] := f4[f4[x]] // Simplify
```

```
λ = 0.8861; FixedPoint[f8, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &)]
```

```
0.365049
```

Hence I estimate $\lambda_3 = 0.88605$

Now we look where the fixed point of f8 disappears. There is a fixed point for $\lambda = 0.89105$

```
λ = 0.89105; FixedPoint[f8, 0.7, SameTest → (Abs[#1 - #2] < 10^-11 &)]
```

```
0.374727
```

but not for $\lambda = 0.89110$ so I estimate $\lambda_4 = 0.89107$.

Now we compute the Feigenbaum constant:

```
la[0] = 0.25; la[1] = 0.75; la[2] = 0.86235; la[3] = 0.88605; la[4] = 0.89107;

Print[k, "       ", δk];
Do[Print[k, "      ", (la[k] - la[k - 1]) / (la[k + 1] - la[k])], {k, 1, 3}]
```

```
0.004       δ0.004
1      4.45038
2      4.74051
3      4.72112
```

These estimates compare fairly well with the correct value of 4.669... .

This is sufficient for the homework. However, for the hell of it, we will now work a bit harder and get these three estimates more accurately and also obtain two additional estimates of δ

```
Clear["Global`*"]
```

It turns out to help *Mathematica* by writing f(x) as follows:

```
f[x_] = 4 λ ((1 / 2)^2 - (x - 1 / 2)^2);

Select[λ /. NSolve[{f[x] == x, f'[x] == 1}, {x, λ}], Im[#] == 0 &]
```

```
{0.25, 0.25}
```

```
l[0] = %[[1]]
```

```
0.25
```

```
Select[λ /. NSolve[{f[x] == x, f'[x] == -1}, {x, λ}],  Im[#] == 0 &]
```

```
{0.75, -0.25}
```

```
l[1] = Select[%, 0 × .25 < # < 1 &][[1]]
```

```
0.75
```

```
f2[x_] = f[f[x]];

Select[λ /. NSolve[{f2[x] == x, f2'[x] == -1}, {x, λ}], Im[#] == 0 &]
```

```
{-0.362372, 0.862372, 0.862372, -0.362372}
```

```
l[2] = Select[%, 0.75 < # < 1 &][[1]]
```

```
0.862372
```

```
f4[x_] = f2[f2[x]] ;
```

```
FindRoot[{f4[x] == x, f4'[x] == -1}, {x, 0.8} , {λ, 0.88}]
```

```
{x → 0.819785, λ → 0.886023}
```

```
l[3] = λ /. %
```

```
0.886023
```

```
f8[x_] = f4[f4[x]];
```

```
FindRoot[{f8[x] == x, f8'[x] == -1}, {x, 0.891}, {λ, 0.89}]
```

```
{x → 0.890787, λ → 0.891102}
```

```
l[4] = λ /. %
```

```
0.891102
```

```
f16[x_] = f8[f8[x]];
```

```
FindRoot[{f16[x] == x, f16'[x] == -1}, {x, 0.892}, {λ, 0.892}]
```

```
{x → 0.89214, λ → 0.89219}
```

```
l[5] = λ /. %
```

```
0.89219
```

```
f32[x_] = f16[f16[x]];
```

```
FindRoot[{f32[x] == x, f32'[x] == -1}, {x, 0.8923}, {λ, 0.8923}]
```

```
{x → 0.892415, λ → 0.892423}
```

```
l[6] = λ /. %
```

```
0.892423
```

```
Print [ k, "       ", δk];
Do[ Print[n , "      ", ( l[n] - l[n - 1]) / (l[n + 1] - l[n])], {n, 1, 5}]
```

```
k       δk
1      4.44949
2      4.75145
3      4.65625
4      4.66824
5      4.66874
```

We see very good convergence to the Feigenbaum ratio 4.6692... .

Qu. 7

- **(a)**

```
f[x_] := λ Sin[Pi x]

lya[l_, xinit_, n_, ndrop_] := (λ = l;
  xlist = Drop[NestList[f, xinit, n], ndrop + 1];
  Apply[Plus, Log[Abs[f'[xlist]]]] / Length[xlist])

lya[0.9, 0.4, 10 000, 1000]
```

```
0.35973
```

The Lyapunov is positive so we are in the chaotic regime.

- **(b)**

Since we are in the chaotic regime we need to use high-precision numbers to get an accurate result for x_{5000}starting from x_0= 4/10.Let's write λ exactly

```
λ = 9 / 10;
```

and make an initial guess that 1000 digits will be enough by spefifying $x_0$to 1000 digits

```
x0 = N[4 / 10, 1000];
```

The we do the iterations and compute the precision.

```
ans = Nest[f, x0, 5000];

Precision[ans]
```

```
251.78
```

This is plenty of precision so we print the result.

```
N[ans]
```

```
0.795585
```

Note that if we use default precision we get a quite wrong answer

```
Nest[f, 0.4, 5000]
```

```
0.899696
```