PHYSICS 150

Homework 6

Due by 11:59 pm, **Wednesday** December 1, 2021

You must explain your work.

1. *Phase Estimation Algorithm*

   Read section 17.4 of the lecture material which describes how to estimate an eigenvalue of a unitary operator U using a quantum circuit.

   It is shown that the eigenvalues are a pure phase, i.e. are of the form $e^{i\theta}$ for some phase θ.

   We take out a factor of 2π and so write $\theta = 2\pi\phi$, where $0 \le \phi < 1$. If we want to determine ϕ to n bits of precision we can write $\phi = \phi'/2^n$ where ϕ' is an integer in the range from 0 to $2^n - 1$.

   Here consider the following two unitary matrices:

$$\text{(a)} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

$$\text{(b)} \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}. \tag{2}$$

   For each matrix determine how many bits n you need to evaluate the eigenvalues, and draw the quantum circuit for each case. Explain how each circuit works.

2. Consider the 3-qubit, bit-flip code discussed in class, and in the lecture material. The circuit is shown in Fig. 1. We commented that this circuit works in the situation where a bit-flip error builds up continuously from zero. Let us verify this. Consider the corrupted state

$$|\psi'\rangle = \left[(1 - \epsilon^2/2)\mathbb{1} + i\,(\,\epsilon_1 X_1 + \epsilon_2 X_2 + \epsilon_3 X_3\,)\right]|\psi\rangle, \tag{3}$$

   where $\epsilon_k \ll 1$ and $\epsilon^2 = \sum_{k=1}^{3} \epsilon_k^2$ and

$$|\psi\rangle = \alpha|000\rangle + \beta|111\rangle \tag{4}$$

   is the uncorrupted state. We will work to first order in ϵ (the factor of $1 - \epsilon^2/2$ is inserted so that the normalization constant is 1 through order ϵ^2). $|\psi'\rangle$ is the initial state (on the left) of the three computational qubits, labeled 1, 2 and 3, in Fig. 1.

   Determine the state of the system (computational qubits plus ancillas) after the error detection circuit has operated.

   Then consider the correction phase. What are the possible results of the measurements of the ancillas, what are the probabilities of these results, and what is the resulting state of the computational qubits?
   (You should conclude that the bit-flip error has been corrected for all possible results of the measurement of the ancillas.)

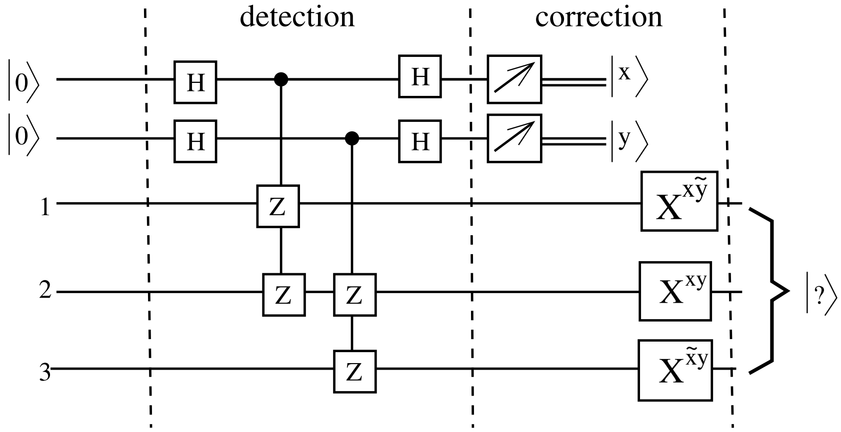

Figure 1: Circuit for syndrome detection for the 3-qubit bit-flip code, and for correction if necessary.

3. In question 2 we implicitly assumed that the time dependence of the computational qubits has proceeded in a unitary manner including the point where the error has developed. In other words the error is in the circuit itself. However, a very common cause of errors in a quantum computer is that the qubits have an unwanted interaction with the environment. The environment becomes entangled with the qubits leading to "decoherence", which is the main difficulty in building a useful quantum computer.

   Let us apply the same 3-qubit, bit-flip code shown in Fig. 1 to an error model where the error comes from the prior interaction of the qubits with the environment.

   A system interacting with environment is not in a single quantum state but can be represented as being in different quantum states with various probabilities[1]. Let us suppose, then, that the system is described as follows (in which we again only allow for single bit-flips):

$$
\begin{aligned}
&\text{Probability}: P_0, \quad |\psi'\rangle = \alpha|000\rangle + \beta|111\rangle = |\psi\rangle \\
&\text{Probability}: P_1, \quad |\psi'\rangle = \alpha|100\rangle + \beta|011\rangle = X_1|\psi\rangle \\
&\text{Probability}: P_2, \quad |\psi'\rangle = \alpha|010\rangle + \beta|101\rangle = X_2|\psi\rangle \\
&\text{Probability}: P_3, \quad |\psi'\rangle = \alpha|001\rangle + \beta|110\rangle = X_3|\psi\rangle,
\end{aligned} \tag{5}
$$

   where, of course, $\sum_{i=0}^{3} P_i = 1$. Note that these states are *incoherent* in the sense that there is no interference between the different states. This is different from Eq. (3) where the different pieces of the wave function have well defined relative phases (i.e. the superposition is *coherent*) and so can potentially interfere.

   Describe the result of acting with the "detection" part of the circuit.

   Then consider the "correction" part and derive the possible results of the measurements of the ancillas and their probabilities. Show that, like the case of the coherent bit-flip error

[1] The correct way to describe this is with the density matrix discussed in Chapter 5, but we will not need the details of the density matrix here.

of Eq. (3) in Qu. 2, the circuit succeeds in correcting the error.

Note: The difference between questions 2 and 3 is that in the former the corruption is due to a *coherent* superposition of 1-qubit corrupted states, while in the latter it is due to an *incoherent* sum of 1-qubit corrupted states with various probabilities. To answer Qu. 3 you have to discuss, *for each of the states in the incoherent sum*, what is the state of the ancillas and how the error correction is done.
By doing both these two questions you see that error correction works irrespective of whether the error is due to a coherent addition of corrupted states (perhaps due to the gates not functioning correctly) or to an incoherent addition of corrupted states due to the computational qubits becoming entangled with the environment.

4. As discussed in class, the four stabilizers for the 5-qubit error correcting code are

$$M_1 = Z_2 X_3 X_4 Z_5, \tag{6a}$$
$$M_2 = Z_3 X_4 X_5 Z_1, \tag{6b}$$
$$M_3 = Z_4 X_5 X_1 Z_2, \tag{6c}$$
$$M_4 = Z_5 X_1 X_2 Z_3. \tag{6d}$$

We also stated that the pattern of $+1$ and -1 eigenvalues for the stabilizers among the 16 syndromes (1 uncorrupted and $3 \times 5 = 15$ corrupted) are given by

	$X_1 Y_1 Z_1$	$X_2 Y_2 Z_2$	$X_3 Y_3 Z_3$	$X_4 Y_4 Z_4$	$X_5 Y_5 Z_5$	$\mathbb{1}$
$M_1 = Z_2 X_3 X_4 Z_5$	$+++$	$--+$	$+--$	$+--$	$--+$	$+$
$M_2 = Z_3 X_4 X_5 Z_1$	$--+$	$+++$	$--+$	$+--$	$+--$	$+$
$M_3 = Z_4 X_5 X_1 Z_2$	$+--$	$--+$	$+++$	$--+$	$+--$	$+$
$M_4 = Z_5 X_1 X_2 Z_3$	$+--$	$+--$	$--+$	$+++$	$--+$	$+$

,

where the top row indicates which Pauli operator is used to generate the corrupted state from the uncorrupted state.

(a) Show that the stabilizers square to the identity.

(b) Show that they are mutually commuting.

(c) By considering the nature of the commutation of the stabilizer with the relevant Pauli operator, explain the results in the table for the columns X_3, Y_4 and Z_5.
Note: You may assume without proof that the right-hand column is correct, i.e. the eigenvalues of all the stabilizers are $+1$ for the uncorrupted state.

5. Using the expressions for the stabilizers of the 5-qubit code given in Qu. 4, draw the circuit to detect 1-qubit errors in the 5-qubit code.

6. *(More challenging)*
Consider the 7-qubit Steane code. There are 6 stabilizers which are

$$\begin{aligned} M_1 &= X_1 X_5 X_6 X_7, & N_1 &= Z_1 Z_5 Z_6 Z_7, \\ M_2 &= X_2 X_4 X_6 X_7, & N_2 &= Z_2 Z_4 Z_6 Z_7, \\ M_3 &= X_3 X_4 X_5 X_7, & N_3 &= Z_3 Z_4 Z_5 Z_7. \end{aligned} \tag{7}$$

The circuit to detect errors is shown in Fig. 2. The 7-qubit codewords are given by

$$
\begin{aligned}
|\overline{0}\rangle &= \frac{1}{\sqrt{8}}(1+M_1)(1+M_2)(1+M_3)|0\rangle_7, \\
|\overline{1}\rangle &= \frac{1}{\sqrt{8}}(1+M_1)(1+M_2)(1+M_3)\overline{X}|0\rangle_7,
\end{aligned} \tag{8}
$$

where "1" refers to the identity operator,

$$\overline{X} = X_1X_2X_3X_4X_5X_6X_7, \tag{9}$$

so

$$|1111111\rangle = \overline{X}|0000000\rangle. \tag{10}$$

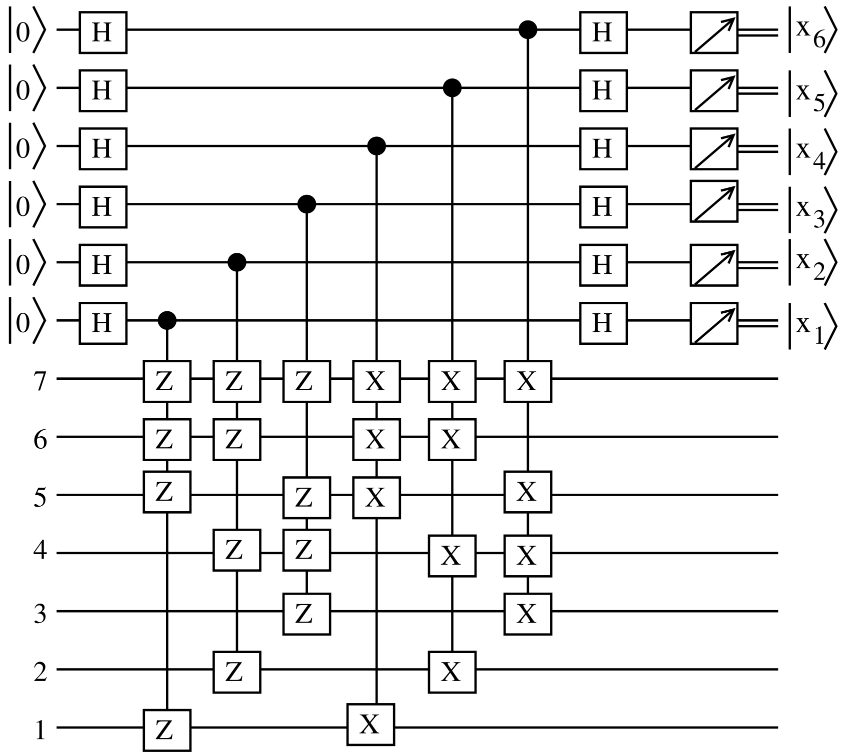

Figure 2: The circuit of Steane's 7-qubit code to detect errors in the computational qubits, (labeled 1–7 in the figure). There are also six ancilla qubits (at the top) each of which is associated with one of the stabilizers as follows: N_1-N_3 correspond to x_1-x_3, and M_1-M_3 correspond to x_4-x_6, in the usual way, e.g. $M_1 = (-1)^{x_1}$.

(a) Show that the stabilizers mutually commute and square to the identity.

(b) Show that the two states in Eq. (8) are orthogonal.

(c) Show that the two states in Eq. (8) are normalized.
Hint: You will need to use that the M_i square to the identity, as does $\overline{X}$, and that $\overline{X}$ commutes with the M_i.

(d) Show that the codewords $|\overline{0}\rangle$ and $|\overline{1}\rangle$ are eigenstates of each of the stabilizers with eigenvalue $+1$.
Hint: Note that $M_i(1+M_i) = 1 + M_i$ (why?), that the N_j commute with $\overline{X}$ (explain why), and that $|0\rangle_7$ is an eigenstate of the N_i with eigenvalue 1.

7. *(More challenging)*
Consider operators which act equally on all qubits in the 7 qubit code:

$$\overline{Z} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7, \qquad \overline{H} = H_1H_2H_3H_4H_5H_6H_7, \tag{11}$$

and similarly $\overline{X}$ defined in Eq. (9).

(a) Show that $\overline{X}$ implements the logical NOT gate (i.e. logical X) on the codewords, i.e.

$$\overline{X}|\overline{0}\rangle = |\overline{1}\rangle, \quad \overline{X}|\overline{1}\rangle = |\overline{0}\rangle. \tag{12}$$

(b) Show that $\overline{Z}$ implements the logical Z on the codewords, i.e.

$$\overline{Z}|\overline{0}\rangle = |\overline{0}\rangle, \quad \overline{Z}|\overline{1}\rangle = -|\overline{1}\rangle. \tag{13}$$

(c) (Harder) Show that $\overline{H}$ implements the logical H on the codewords, i.e.

$$\overline{H}|\overline{0}\rangle = \frac{1}{\sqrt{2}}\left(|\overline{0}\rangle + |\overline{1}\rangle\right), \quad \overline{H}|\overline{1}\rangle = \frac{1}{\sqrt{2}}\left(|\overline{0}\rangle - |\overline{1}\rangle\right). \tag{14}$$

Hints:

- We want to show that

$$\langle\overline{0}|\overline{H}|\overline{0}\rangle = \langle\overline{1}|\overline{H}|\overline{0}\rangle = \langle\overline{0}|\overline{H}|\overline{1}\rangle = \frac{1}{\sqrt{2}}, \quad \langle\overline{1}|\overline{H}|\overline{1}\rangle = -\frac{1}{\sqrt{2}}. \tag{15}$$

- Hence we need to calculate

$$\begin{aligned}\langle\overline{x}|\overline{H}|\overline{y}\rangle = \frac{1}{8}\,{}_7\langle 0|\overline{X}^x(1+M_1)(1+M_2)(1+M_3)\overline{H}(1+M_1)& \\ \times (1+M_2)(1+M_3)\overline{X}^y|0\rangle_7.&\end{aligned} \tag{16}$$

- Derive the results

$$\overline{H}M_i = N_i\overline{H}, \quad M_i\overline{H} = \overline{H}N_i, \tag{17}$$

and use them to show that you can replace the M_i in Eq. (16) by N_i.
- Show that each N_i commutes with $\overline{X}$ and apply this result.
- Use that each N_i acts as the identity on $|0\rangle_7$.

Note: Having codeword gates that are tensor products of single qubit gates is very helpful when designing circuits to implement an error correcting code. A similar result also holds for CNOT. In Steane's code the logical CNOT gate that takes $|\overline{x}\rangle|\overline{y}\rangle$ to $|\overline{x}\rangle|\overline{x \oplus y}\rangle$, is simply made up of CNOT gates applied to each of the seven pairs of qubits in the two codewords.

The results in this question for Hadamards and CNOT gates do not apply, for example, to the 5 qubit code of Qu. 4. That they do apply to Steane's 7 qubit code is one of the reasons why this code is a popular choice.

8. In the last question we showed that, for the 7-qubit Steane code, the logical $\overline{X}$ acting on the codewords is implemented by $\prod_j X_j$, and the logical $\overline{Z}$ is implemented by $\prod_j Z_j$. Show that the corresponding results for Shor's 9-qubit code do not hold. Instead, show that one has, rather curiously,

$$\prod_{j=1}^{9} Z_j \equiv \overline{X}, \qquad \prod_{j=1}^{9} X_j \equiv \overline{Z}. \tag{18}$$