The purpose of this handout is to clarify the distinction between a coherent superposition of amplitudes in quantum mechanics and an incoherent (classical) addition of probabilities.

I. COHERENT LINEAR SUPERPOSITION: 1 QUBIT

To illustrate coherent superposition, consider one qubit in the following state

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \tag{1} \]

where \(|\alpha|^2 + |\beta|^2 = 1\). We will denote \(|\alpha|^2\) by \(p\). Evidently \(|\psi\rangle\) is a linear superposition of basis states \(|0\rangle\) and \(|1\rangle\). We say it is a coherent superposition because there is a well defined phase relationship between the pieces in the superposition, which means that there can be interference between these pieces in subsequent operations.

If we measure \(|\psi\rangle\) in the computational basis we get

\[ |0\rangle \text{ with probability } |\alpha|^2 = p, \]
\[ |1\rangle \text{ with probability } |\beta|^2 = 1 - p. \tag{2} \]

To show the effects of interference we now apply a Hadamard gate before doing the measurement. The result is

\[ |\psi'\rangle = H|\psi\rangle = \frac{\alpha}{\sqrt{2}} (|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}} (|0\rangle - |1\rangle) = \left( \frac{\alpha + \beta}{\sqrt{2}} \right) |0\rangle + \left( \frac{\alpha - \beta}{\sqrt{2}} \right) |1\rangle. \tag{3} \]

If we now do a measurement in the computational basis the results are

\[ |0\rangle \text{ with probability } \frac{1}{2} |\alpha + \beta|^2 = \frac{1}{2} (1 + \alpha^* \beta + \alpha \beta^*), \]
\[ |1\rangle \text{ with probability } \frac{1}{2} |\alpha - \beta|^2 = \frac{1}{2} (1 - \alpha^* \beta - \alpha \beta^*). \tag{4} \]

The factor \(\alpha \beta^* + \alpha^* \beta\) comes from interference between the two pieces in the linear combination of \(|\psi\rangle\) in Eq. (1). In particular, if \(\alpha = \beta = \frac{1}{\sqrt{2}}\), so \(p = \frac{1}{2}\), we get

\[ |0\rangle \text{ with probability } 1, \]
\[ |1\rangle \text{ with probability } 0. \tag{5} \]
showing that there is zero probability of getting state $|1\rangle$ for $\alpha = \beta = \frac{1}{\sqrt{2}}$ if we measure after performing a Hadamard. The vanishing probability of getting $|1\rangle$ is due to destructive interference between the two pieces of the superposition in state $|\psi\rangle$ in Eq. (1).

II. INCOHERENT (CLASSICAL) ADDITION OF PROBABILITIES: 2 QUBITS

To illustrate the incoherent addition of probabilities consider two qubits in the following entangled state

$$|\psi_2\rangle = \alpha|00\rangle + \beta|11\rangle,$$

where we will again denote $|\alpha|^2$ by $p$. If $\alpha = \pm \beta = \frac{1}{\sqrt{2}}$ this is a Bell state. Let us write $|\psi_2\rangle$ more explicitly as

$$|\psi_2\rangle = \alpha|0_A\rangle \otimes |0_B\rangle + \beta|1_A\rangle \otimes |1_B\rangle.$$  (7)

If we focus on qubit $A$, say, then state $|\psi_2\rangle$ looks rather similar to the 1-qubit state $|\psi\rangle$ in Eq. (1), in that there is a piece where qubit $A$ is $|0\rangle$ with amplitude $\alpha$ and a piece where qubit $A$ is $|1\rangle$ with amplitude $\beta$. However, for $|\psi_2\rangle$, unlike for $|\psi\rangle$, each of these pieces goes with a different state for qubit $B$ (because $|\psi_2\rangle$ is is entangled). As a result, we will not get interference between the pieces of $|\psi_2\rangle$ if we perform operations on qubit $A$ followed by a measurement of that qubit.

We can focus on the behavior of qubit $A$ by computing its density matrix, see the handout https://young.physics.ucsc.edu/150/density_matrix.pdf. Writing

$$|\psi_2\rangle = \sum_{i,j=0}^{1} c_{ij} |i_A\rangle \otimes |j_B\rangle$$  (8)

we have here $c_{00} = \alpha$, $c_{11} = \beta$, $c_{01} = c_{10} = 0$. As shown in the handout, the elements of the density matrix for $A$ are given by

$$\rho^A_{i,i'} = \sum_{j=0}^{1} c_{ij} c^*_{i'j}$$  (9)

so here $\rho^A_{00} = c_{00}c^*_{00} = |\alpha|^2 = p$, $\rho^A_{11} = c_{11}c^*_{11} = |\beta|^2 = 1 - p$ and $\rho^A_{01} = \rho^A_{10} = 0$. Thus we have

$$\rho^A = \begin{pmatrix} p & 0 \\ 0 & 1 - p \end{pmatrix}.$$  (10)

Trivially, the eigenvalues are $p$ and $1 - p$ with corresponding eigenvectors $|0\rangle$ and $|1\rangle$. As discussed in the handout on the density matrix, this means that if we focus on qubit $A$, performing
unitary operations and measurements just on this qubit, then the qubit can be regarded as initially
being in state $|0\rangle$ with probability $p$ and in state $|1\rangle$ with probability $1 - p$.

If we measure qubit $A$ before doing any operation on it we get

$$
|0\rangle \text{ with probability } p,
|1\rangle \text{ with probability } 1 - p,
$$

which is the same as in Eq. (2) for a single qubit in a coherent superposition.

However, a difference appears if we perform a unitary transformation on qubit $A$ before mea-
suring it. Here we apply a Hadamard as we did in Sec. I. Before acting with $H$, the state of qubit
$A$ is described by the density matrix in Eq. (10), which means, as stated above, that qubit $A$ is
in state $|0\rangle$ with probability $p$ and in state $|1\rangle$ with probability $1 - p$. Hence, after acting with $H$, qubit $A$ is\(^1\)

$$
in state \ H|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \text{ with probability } p,
$$

$$
in state \ H|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \text{ with probability } 1 - p.
$$

We then measure in the computational basis. If qubit $A$ is in state $H|0\rangle$ (which occurs with
probability $p$) one has probability $\frac{1}{2}$ to get $|0\rangle$ and probability $\frac{1}{2}$ to get $|1\rangle$. If qubit $A$ is in state
$H|1\rangle$ (which occurs with probability $1 - p$) one again has probability $\frac{1}{2}$ to get $|0\rangle$ and probability
$\frac{1}{2}$ to get $|1\rangle$. Combining these possible outcomes, one obtains from a measurement of qubit $A$:

$$
|0\rangle \text{ with probability } \frac{1}{2}(p + 1 - p) = \frac{1}{2},
|1\rangle \text{ with probability } \frac{1}{2}(p + 1 - p) = \frac{1}{2},
$$

which is independent of $p$. Equation (13) differs from Eq. (4), the case of a coherent superposition,
by the absence of the factors of $\alpha\beta^* + \alpha^*\beta$ which come from interference.

To summarize, for a coherent superposition one sums the amplitudes and then squares, e.g.

$$
\frac{1}{2} |\alpha + \beta|^2,
$$

while for an incoherent superposition one squares and then sums, e.g.

$$
\frac{1}{2} (|\alpha|^2 + |\beta|^2).
$$

\(^1\) We will verify this in Appendix A by directly computing the new density matrix after $H$ has acted on qubit $A$.  

Appendix A: Computation of the Density Matrix $\rho^A$ after the action of the Hadamard

Here we verify that Eq. (12) is correct by working out from scratch the density matrix for the state

$$ |\psi_2'\rangle = H_A |\psi_2\rangle 
= \alpha (H_A |0_A\rangle) \otimes |0_B\rangle + \beta (H_A |1_A\rangle) \otimes |1_B\rangle $$

(A1)

$$ = \frac{\alpha}{\sqrt{2}} (|0_A0_B\rangle + |1_A0_B\rangle) + \frac{\beta}{\sqrt{2}} (|0_A1_B\rangle - |1_A1_B\rangle). $$

The coefficients are

$$ c_{00} = \alpha \sqrt{2}, \quad c_{10} = \frac{\alpha}{\sqrt{2}}, \quad c_{01} = \beta \sqrt{2}, \quad c_{11} = -\frac{\beta}{\sqrt{2}}, $$

(A2)

so the elements of the density matrix $\rho^A$ are given by

$$ \rho^A_{00} = c_{00}c_{00} + c_{01}c_{01} = \frac{1}{2} (|\alpha|^2 + |\beta|^2) = \frac{1}{2} $$

$$ \rho^A_{10} = c_{00}c_{10} + c_{01}c_{11} = \frac{1}{2} (|\alpha|^2 - |\beta|^2) = \frac{1}{2}(2p - 1) $$

$$ \rho^A_{11} = c_{10}c_{00} + c_{11}c_{01} = \frac{1}{2} (|\alpha|^2 + |\beta|^2) = \frac{1}{2}, $$

(A3)

and hence

$$ \rho^A = \frac{1}{2} \begin{pmatrix} 1 & 2p - 1 \\ 2p - 1 & 1 \end{pmatrix}. $$

(A4)

The eigenvalues are given by

$$ \begin{vmatrix} \frac{1}{2} - \lambda & p - \frac{1}{2} \\ p - \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} $$

(A5)

which gives

$$ (\lambda - \frac{1}{2})^2 - (p - \frac{1}{2})^2 = 0, $$

(A6)

which can be written as

$$ \lambda^2 - \lambda + p - p^2 = (\lambda - p)(\lambda - 1 + p) = 0, $$

(A7)

so the solutions are

$$ \lambda = p \text{ and } 1 - p. $$

(A8)
For $\lambda = p$ one finds that the eigenvector is $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, while for $\lambda = 1 - p$ one finds that the eigenvector is $\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$.

Hence, with probability $p$, qubit $A$ is in state $\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, and with probability $1 - p$ the qubit is in state $\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$, exactly as given in Eq. (12).

More properties of the density matrix are discussed in Appendix B

**Appendix B: Change in the density matrix under a unitary transformation**

If qubit $A$ (more generally subsystem $A$) is acted by a unitary transformation $U^A$ then we show now that the density matrix for subsystem $A$ changes from $\rho^A$ to $\rho'^A$ where:

$$\rho'^A = U^A \rho^A (U^A)^\dagger.$$  \hfill (B1)

To see this, note that $|\psi_2\rangle$ in Eq. (7) goes to $|\psi'_2\rangle$ where

$$|\psi'_2\rangle = \sum_{i,j} c'_{ij} |i_A\rangle \otimes |j_B\rangle$$ \hfill (B2)

in which

$$c'_{ij} = \sum_k U^A_{ik} c_{kj}$$ \hfill (B3)

describes the change in amplitudes produced by the action of $U^A$. Note that the second index $j$ on the amplitude $c_{ij}$ refers to subsystem $B$ and is not changed. Hence

$$\rho'^A_{i,i'} = \sum_j c'_{ij} c'^*_{i'j} \quad = \sum_{j,k_1,k_2} U^A_{ik_1} c_{k_1j} U^A_{i'k_2} c'^*_{k_2j}$$

$$\quad = \sum_{k_1,k_2} U^A_{ik_1} \left( \sum_j c_{k_1j} c'^*_{k_2j} \right) U^A_{i'k_2}$$

$$\quad = \sum_{k_1,k_2} U^A_{ik_1} \rho^A_{k_1,k_2} (U^A_{k_2i'})^\dagger$$

$$\quad = \left( (U^A \rho^A (U^A)^\dagger)_{i,i'} \right),$$

so we obtain Eq. (B1).

For example with $\rho^A$ given by Eq. (10), and $U^A = (U^A)^\dagger = H_A$, a Hadamard, we have

$$\rho'^A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 - p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 2p - 1 \\ 2p - 1 & 1 \end{pmatrix}.$$  \hfill (B5)
which agrees with Eq. (A4).

Note that the most general operation that can be applied to the combined $AB$ system is a unitary transformation acting on the whole system, not just on subsystem $A$ as discussed in this handout so far. One can show that if one performs such a general unitary operation on the combined system, and then recomputes the density matrix of subsystem $A$, the new density matrix is not necessarily related to the old one by a unitary transformation. This is how irreversible processes can occur in a subsystem when it is coupled to a system with a very large number of degrees of freedom, such as the environment. A more detailed discussion of this is beyond the scope of the course but the interested student is referred to Nielsen and Chuang[1] and Rieffel and Polak[2].
