Coherent Superposition Versus Incoherent Addition of Probabilities

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The purpose of this handout is to clarify the distinction between a *coherent* superposition of *amplitudes* in quantum mechanics and an *incoherent* (classical) addition of *probabilities*.

I. COHERENT LINEAR SUPERPOSITION: 1 QUBIT

To illustrate coherent superposition, consider one qubit in the following state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{1}$$

where $|\alpha|^2 + |\beta|^2 = 1$. We will denote $|\alpha^2|$ by p. Evidently $|\psi\rangle$ is a linear superposition of basis states $|0\rangle$ and $|1\rangle$. We say it is a *coherent* superposition because there is a well defined phase relationship between the pieces in the superposition, which means that there can be *interference* between these pieces in subsequent operations.

If we measure $|\psi\rangle$ in the computational basis we get

$$\begin{aligned} &|0\rangle \text{ with probability } |\alpha|^2 = p, \\ &|1\rangle \text{ with probability } |\beta|^2 = 1-p. \end{aligned} \tag{2}$$

To show the effects of interference we now apply a Hadamard gate before doing the measurement. The result is

$$|\psi'\rangle = H|\psi\rangle = \frac{\alpha}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \left(\frac{\alpha+\beta}{\sqrt{2}}\right)|0\rangle + \left(\frac{\alpha-\beta}{\sqrt{2}}\right)|1\rangle. \tag{3}$$

If we *now* do a measurement in the computational basis the results are

$$\begin{aligned} &|0\rangle \text{ with probability } \tfrac{1}{2}|\alpha+\beta|^2 = \tfrac{1}{2}\left(1 + \alpha\beta^* + \alpha^*\beta\right), \\ &|1\rangle \text{ with probability } \tfrac{1}{2}|\alpha-\beta|^2 = \tfrac{1}{2}\left(1 - \alpha\beta^* - \alpha^*\beta\right). \end{aligned} \tag{4}$$

The factor $\alpha\beta^* + \alpha^*\beta$ comes from *interference* between the two pieces in the linear combination of $|\psi\rangle$ in Eq. (1). In particular, if $\alpha = \beta = \frac{1}{\sqrt{2}}$, so $p = \frac{1}{2}$, we get

$$\begin{aligned} &|0\rangle \text{ with probability } 1, \\ &|1\rangle \text{ with probability } 0, \end{aligned} \tag{5}$$

showing that there is *zero* probability of getting state $|1\rangle$ for $\alpha = \beta = \frac{1}{\sqrt{2}}$ if we measure after performing a Hadamard. The vanishing probability of getting $|1\rangle$ is due to destructive interference between the two pieces of the superposition in state $|\psi\rangle$ in Eq. (1).

II. INCOHERENT (CLASSICAL) ADDITION OF PROBABILITIES: 2 QUBITS

To illustrate the incoherent addition of probabilities consider two qubits in the following entangled state

$$|\psi_2\rangle = \alpha|00\rangle + \beta|11\rangle, \tag{6}$$

where we will again denote $|\alpha|^2$ by p. If $\alpha = \pm\beta = \frac{1}{\sqrt{2}}$ this is a Bell state. Let us write $|\psi_2\rangle$ more explicitly as

$$|\psi_2\rangle = \alpha|0_A\rangle \otimes |0_B\rangle + \beta|1_A\rangle \otimes |1_B\rangle. \tag{7}$$

If we focus on qubit A, say, then state $|\psi_2\rangle$ looks rather similar to the 1-qubit state $|\psi\rangle$ in Eq. (1), in that there is a piece where qubit A is $|0\rangle$ with amplitude α and a piece where qubit A is $|1\rangle$ with amplitude β. However, for $|\psi_2\rangle$, unlike for $|\psi\rangle$, each of these pieces goes with a different state for qubit B (because $|\psi_2\rangle$ is is entangled). As a result, we will *not* get interference between the pieces of $|\psi_2\rangle$ if we perform operations on qubit A followed by a measurement of that qubit.

We can focus on the behavior of qubit A by computing its density matrix, see the handout `https://young.physics.ucsc.edu/150/density_matrix.pdf`. Writing

$$|\psi_2\rangle = \sum_{i,j=0}^{1} c_{ij}|i_A\rangle \otimes |j_B\rangle \tag{8}$$

we have here $c_{00} = \alpha, c_{11} = \beta, c_{01} = c_{10} = 0$. As shown in the handout, the elements of the density matrix for A are given by

$$\rho^A_{i,i'} = \sum_{j=0}^{1} c_{ij}c^*_{i'j} \tag{9}$$

so here $\rho^A_{00} = c_{00}c^*_{00} = |\alpha|^2 = p, \rho^A_{11} = c_{11}c^*_{11} = |\beta|^2 = 1-p$ and $\rho^A_{01} = \rho^A_{10} = 0$. Thus we have

$$\rho^A = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}. \tag{10}$$

Trivially, the eigenvalues are p and $1-p$ with corresponding eigenvectors $|0\rangle$ and $|1\rangle$. As discussed in the handout on the density matrix, this means that if we focus on qubit A, performing

unitary operations and measurements just on this qubit, then the qubit can be regarded as initially being in state $|0\rangle$ with probability p and in state $|1\rangle$ with probability $1-p$.

If we measure qubit A before doing any operation on it we get

$$\begin{aligned} &|0\rangle \text{ with probability } p, \\ &|1\rangle \text{ with probability } 1-p, \end{aligned} \tag{11}$$

which is the same as in Eq. (2) for a single qubit in a coherent superposition.

However, a difference appears if we perform a unitary transformation on qubit A before measuring it. Here we apply a Hadamard as we did in Sec. I. Before acting with H, the state of qubit A is described by the density matrix in Eq. (10), which means, as stated above, that qubit A is in state $|0\rangle$ with probability p and in state $|1\rangle$ with probability $1-p$. Hence, after acting with H, qubit A is[1]

$$\begin{aligned} &\text{in state } H|0\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \text{with probability } p, \\ &\text{in state } H|1\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \text{with probability } 1-p. \end{aligned} \tag{12}$$

We then measure in the computational basis. If qubit A is in state $H|0\rangle$ (which occurs with probability p) one has probability $\frac{1}{2}$ to get $|0\rangle$ and probability $\frac{1}{2}$ to get $|1\rangle$. If qubit A is in state $H|1\rangle$ (which occurs with probability $1-p$) one again has probability $\frac{1}{2}$ to get $|0\rangle$ and probability $\frac{1}{2}$ to get $|1\rangle$. Combining these possible outcomes, one obtains from a measurement of qubit A:

$$\begin{aligned} &|0\rangle \text{ with probability } \tfrac{1}{2}(p+1-p) = \tfrac{1}{2}, \\ &|1\rangle \text{ with probability } \tfrac{1}{2}(p+1-p) = \tfrac{1}{2}, \end{aligned} \tag{13}$$

which is independent of p. Equation (13) differs from Eq. (4), the case of a coherent superposition, by the absence of the factors of $\alpha\beta^* + \alpha^*\beta$ which come from interference.

To summarize, for a coherent superposition one sums the amplitudes and then squares, e.g.

$$\frac{1}{2}|\alpha + \beta|^2, \tag{14}$$

while for an incoherent superposition one squares and then sums, e.g.

$$\frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right). \tag{15}$$

[1] We will verify this in Appendix A by directly computing the new density matrix after H has acted on qubit A.

Appendix A: Computation of the Density Matrix ρ^A after the action of the Hadamard

Here we verify that Eq. (12) is correct by working out from scratch the density matrix for the state

$$\begin{aligned} |\psi_2'\rangle &= H_A|\psi_2\rangle \\ &= \alpha\left(H_A|0_A\rangle\right)\otimes|0_B\rangle + \beta\left(H_A|1_A\rangle\right)\otimes|1_B\rangle \\ &= \frac{\alpha}{\sqrt{2}}\left(|0_A0_B\rangle + |1_A0_B\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|0_A1_B\rangle - |1_A1_B\rangle\right). \end{aligned} \tag{A1}$$

The coefficients are

$$c_{00} = \frac{\alpha}{\sqrt{2}}, \quad c_{10} = \frac{\alpha}{\sqrt{2}}, \quad c_{01} = \frac{\beta}{\sqrt{2}}, \quad c_{11} = -\frac{\beta}{\sqrt{2}}. \tag{A2}$$

so the elements of the density matrix $\rho^{'A}$ are given by

$$\begin{aligned} \rho_{00}^{'A} &= c_{00}c_{00} + c_{01}c_{01} = \tfrac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) = \tfrac{1}{2} \\ \rho_{01}^{'A} &= c_{00}c_{10} + c_{01}c_{11} = \tfrac{1}{2}\left(|\alpha|^2 - |\beta|^2\right) = \tfrac{1}{2}(2p-1) \\ \rho_{10}^{'A} &= c_{10}c_{00} + c_{11}c_{01} = \tfrac{1}{2}\left(|\alpha|^2 - |\beta|^2\right) = \tfrac{1}{2}(2p-1) \\ \rho_{11}^{'A} &= c_{10}c_{10} + c_{11}c_{11} = \tfrac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) = \tfrac{1}{2}, \end{aligned} \tag{A3}$$

and hence

$$\rho^{'A} = \frac{1}{2}\begin{pmatrix} 1 & 2p-1 \\ 2p-1 & 1 \end{pmatrix}. \tag{A4}$$

The eigenvalues are given by

$$\begin{vmatrix} \frac{1}{2}-\lambda & p-\frac{1}{2} \\ p-\frac{1}{2} & \frac{1}{2}-\lambda \end{vmatrix} \tag{A5}$$

which gives

$$\left(\lambda - \tfrac{1}{2}\right)^2 - \left(p - \tfrac{1}{2}\right)^2 = 0, \tag{A6}$$

which can be written as

$$\lambda^2 - \lambda + p - p^2 = (\lambda - p)\,(\lambda - 1 + p) = 0, \tag{A7}$$

so the solutions are

$$\lambda = p \text{ and } 1-p. \tag{A8}$$

For $\lambda = p$ one finds that the eigenvector is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, while for $\lambda = 1-p$ one finds that the eigenvector is $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Hence, with probability p, qubit A is in state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and with probability $1-p$ the qubit is in state $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, exactly as given in Eq. (12).

More properties of the density matrix are discussed in Appendix B

Appendix B: Change in the density matrix under a unitary transformation

If qubit A (more generally subsystem A) is acted by a unitary transformation U^A then we show now that the density matrix for subsystem A changes from ρ^A to $\rho^{'A}$ where:

$$\rho^{'A} = U^A \rho^A \left(U^A\right)^\dagger . \tag{B1}$$

To see this, note that $|\psi_2\rangle$ in Eq. (7) goes to $|\psi_2'\rangle$ where

$$|\psi_2'\rangle = \sum_{i,j} c'_{ij} |i_A\rangle \otimes |j_B\rangle \tag{B2}$$

in which

$$c'_{ij} = \sum_k U^A_{ik}\, c_{kj} \tag{B3}$$

describes the change in amplitudes produced by the action of U^A. Note that the second index j on the amplitude c_{ij} refers to subsystem B and is not changed. Hence

$$\begin{aligned}
\rho^{'A}_{i,i'} &= \sum_j c'_{ij} c^{'*}_{i'j} \\
&= \sum_{j,k_1,k_2} U^A_{ik_1}\, c_{k_1 j}\, U^{A*}_{i'k_2} c^*_{k_2 j} \\
&= \sum_{k_1,k_2} U^A_{ik_1} \left(\sum_j c_{k_1 j} c^*_{k_2 j} \right) U^{A*}_{i'k_2} \\
&= \sum_{k_1,k_2} U^A_{ik_1} \rho^A_{k_1,k_2} \left(U^A_{k_2 i'}\right)^\dagger \\
&= \left(U^A \rho^A \left(U^A\right)^\dagger \right)_{i,i'} ,
\end{aligned} \tag{B4}$$

so we obtain Eq. (B1).

For example with ρ^A given by Eq. (10), and $U^A = \left(U^A\right)^\dagger = H_A$, a Hadamard, we have

$$\rho^{'A} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 2p-1 \\ 2p-1 & 1 \end{pmatrix} \tag{B5}$$

which agrees with Eq. (A4).

Note that the most general operation that can be applied to the combined AB system is a unitary transformation acting on the *whole system*, not just on subsystem A as discussed in this handout so far. One can show that if one performs such a *general* unitary operation on the combined system, and then recomputes the density matrix of subsystem A, the new density matrix is *not necessarily related to the old one by a unitary transformation*. This is how irreversible processes can occur in a subsystem when it is coupled to a system with a very large number of degrees of freedom, such as the environment. A more detailed discussion of this is beyond the scope of the course but the interested student is referred to Nielsen and Chuang[1] and Rieffel and Polak[2].

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[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[2] E. Rieffel and I. Polak, *Quantum Computing; A Gentle Introduction* (MIT Press, Massachusetts Institute of Technology, 2014).