Generating and measuring Bell States

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Entangled states play an important role in quantum computing. The most-studied entangled states are so-called Bell states which involve two qubits. They are named in honor of the physicist who clarified the Einstein-Podolsky-Rosen (EPR) paradox, and whose inequalities demonstrated that the description of nature provided by quantum mechanics is fundamentally different from the classical description. The Bell states are defined by

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(\,|00\rangle + |11\rangle\,\right), \tag{1a}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\left(\,|01\rangle + |10\rangle\,\right), \tag{1b}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}\left(\,|00\rangle - |11\rangle\,\right), \tag{1c}$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\left(\,|01\rangle - |10\rangle\,\right). \tag{1d}$$

These four equations can be combined as follows:

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}\left(\,|0y\rangle + (-1)^x|1\overline{y}\rangle\,\right)\,, \tag{2}$$

where $\overline{y}$ is the complement of y, i.e. $\overline{y} = 1 - y$.

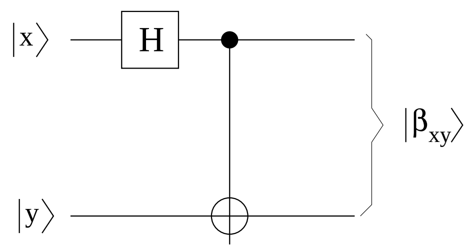

FIG. 1: Circuit to create the Bell states defined by Eqs. (1). In the CNOT gate the upper qubit $|x\rangle$ is the control qubit and the lower qubit $|y\rangle$ is the target qubit.

The Bell states are clearly entangled. They can be created out of two (unentangled) qubits in computational basis states $|xy\rangle$ by the circuit shown in Fig. 1. To see this note that after the Hadamard the state is

$$|xy\rangle \to \frac{1}{\sqrt{2}}\left(\,|0y\rangle + (-1)^x|1y\rangle\,\right)\,. \tag{3}$$

The effect of the CNOT gate is to flip y in the second term (since $x = 1$ there) and so we get Eq. (2)

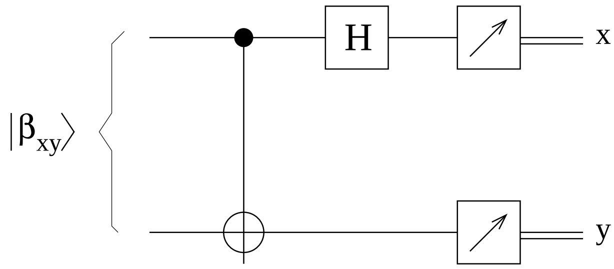

FIG. 2: Circuit for Bell measurements. This will be used later in the course when we discuss teleportation.

The circuit in Fig. 1 converts the computational basis to the Bell basis. The reverse of this circuit can be used to convert the Bell basis back to the computational basis as shown in Fig. 2. The measured values of x and y tell us which Bell state we started with. This is called a *Bell Measurement*. To see that this works note that after the CNOT gate the state of the two qubits in Fig. 2 is[1]

$$\frac{1}{\sqrt{2}}\left[\,|0y\rangle + (-1)^x|1y\rangle\,\right], \tag{4}$$

which is separable and so can be written as

$$\frac{1}{\sqrt{2}}\left[\,|0\rangle + (-1)^x|1\rangle\,\right] \otimes |y\rangle. \tag{5}$$

Recall that the left-hand qubit is the upper (control) qubit in Fig. 2 and the right hand qubit is the lower (target) qubit. Acting with the Hadamard has the effect

$$H\frac{1}{\sqrt{2}}\left[\,|0\rangle + (-1)^x|1\rangle\,\right] = |x\rangle\,, \tag{6}$$

so the final state in Fig. 2 is $|xy\rangle$ as desired.

Note that the Bell states $|\beta_{xy}\rangle$ provide a basis for two qubits, since they are normalized, mutually orthogonal and linearly independent. Consequently, if the state inputted into the Bell measurement circuit in Fig. 2 is not a single Bell state, but rather a linear combination,

$$|\psi_{\rm in}\rangle = \sum_{x,y=0}^{1} \alpha_{xy}|\beta_{xy}\rangle, \tag{7}$$

with $\sum_{x,y}|\alpha_{xy}|^2 = 1$, then the probability that the measurements obtain a particular set of values for x and y is $|\alpha_{xy}|^2$.

[1] The reason that $\overline{y}$ in the Bell state, Eq. (2), changes to y in the second term in Eq. (4) is because $x = 1$ and so the y (target) qubit is flipped.