Generating and measuring Bell States

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Entangled states play an important role in quantum computing. The most-studied entangled states are so-called Bell states which involve two qubits. They are named in honor of the physicist who clarified the Einstein-Podolsky-Rosen (EPR) paradox, and whose inequalities demonstrated that the description of nature provided by quantum mechanics is fundamentally different from the classical description. The Bell states are defined by

\[
|\beta_{00}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle ),
\]

(1a)

\[
|\beta_{01}\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle ),
\]

(1b)

\[
|\beta_{10}\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle ),
\]

(1c)

\[
|\beta_{11}\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle ).
\]

(1d)

These four equations can be combined as follows:

\[
|\beta_{xy}\rangle = \frac{1}{\sqrt{2}} (|0y\rangle + (-1)^x|1\overline{y}\rangle ),
\]

(2)

where \(\overline{y}\) is the complement of \(y\), i.e. \(\overline{y} = 1 - y\).

\[ |x\rangle \quad \text{H} \quad \text{CNOT} \quad |y\rangle \rightarrow |\beta_{xy}\rangle \]

FIG. 1: Circuit to create the Bell states defined by Eqs. (1). In the CNOT gate the upper qubit \(|x\rangle\) is the control qubit and the lower qubit \(|y\rangle\) is the target qubit.

The Bell states are clearly entangled. They can be created out of two (unentangled) qubits in computational basis states \(|xy\rangle\) by the circuit shown in Fig. 1. To see this note that after the Hadamard the state is

\[
|xy\rangle \rightarrow \frac{1}{\sqrt{2}} (|0y\rangle + (-1)^x|1\overline{y}\rangle ).
\]

(3)
The effect of the CNOT gate is to flip $y$ in the second term (since $x = 1$ there) and so we get Eq. (2)

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}} \left[ |0y\rangle + (-1)^x |1y\rangle \right], \quad (4)$$

which is separable and so can be written as

$$\frac{1}{\sqrt{2}} \left[ |0\rangle + (-1)^x |1\rangle \right] \otimes |y\rangle. \quad (5)$$

Recall that the left-hand qubit is the upper (control) qubit in Fig. 2 and the right hand qubit is the lower (target) qubit. Acting with the Hadamard has the effect

$$H \frac{1}{\sqrt{2}} \left[ |0\rangle + (-1)^x |1\rangle \right] = |x\rangle, \quad (6)$$

so the final state in Fig. 2 is $|xy\rangle$ as desired.

Note that the Bell states $|\beta_{xy}\rangle$ provide a basis for two qubits, since they are normalized, mutually orthogonal and linearly independent. Consequently, if the state inputted into the Bell measurement circuit in Fig. 2 is not a single Bell state, but rather a linear combination,

$$|\psi_{\text{in}}\rangle = \sum_{x,y=0}^{1} \alpha_{xy} |\beta_{xy}\rangle, \quad (7)$$

with $\sum_{x,y} |\alpha_{xy}|^2 = 1$, then the probability that the measurements obtain a particular set of values for $x$ and $y$ is $|\alpha_{xy}|^2$.

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1 The reason that $y$ in the Bell state, Eq. (2), changes to $y$ in the second term in Eq. (4) is because $x = 1$ and so the $y$ (target) qubit is flipped.