PHYSICS 116C
Homework 8 Solutions

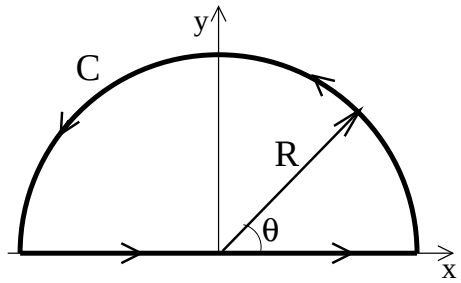
1. We have to compute

$$g(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\, dx.$$

(A similar problem is done in Boas, Example 3, Ch. 14, p. 689).)

We will do this by an appropriate contour integral of $e^{ikz}/(1+z^2)$. This function has simple poles at $z = i$ and $-i$ with residues $e^{-k}/2i$ and $-e^{k}/2i$ respectively.

Now $\exp[ik(x+iy)] = \exp(ikx)\,\exp(-ky)$. We do not want this to blow up so, for $k > 0$, we need to complete the contour in the upper half-plane (where $y > 0$), and for $k < 0$ we complete it in the lower half plane.



The figure above shows the semicircular contour for $k > 0$. We need to take the limit $R \to \infty$. Jordan's lemma (unfortunately not discussed explicitly in Boas) states that if $f(z) \to 0$ for $|z| \to \infty$ then, evaluating the integral of $f(z)e^{ikz}$ around the semicircular contour in the figure for $k > 0$, the contribution from the semi-circle vanishes for $R \to \infty$. (There is an analogous theorem for $k < 0$ in which the semicircle is in the lower half-plane.) Here $f(z) = 1/(1+z^2)$ does vanish at infinity and so (for $k > 0$)

$$\begin{aligned} g(k) &= \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2}\, dx \\ &= \lim_{R\to\infty} \oint_C \frac{e^{ikz}}{1+z^2}\, dz \\ &= 2\pi i\, \frac{e^{-k}}{2i} \\ &= \pi e^{-k}\,, \quad (k > 0), \end{aligned}$$

where we used the residue theorem to get the third line.

For $k < 0$, we complete the contour by a semicircle in the lower half-plane. This gives an extra minus sign, since we traverse the contour clockwise, and the residue is $-e^{k}/2i$. Hence we obtain

$$g(k) = -2\pi i\, \frac{\left(-e^{k}\right)}{2i} = \pi e^{k}\,, \quad (k < 0).$$

The results for the two different signs of k can be combined as

$$\boxed{g(k) = \pi e^{-|k|}\,.}$$

2. Fourier transforming the diffusion equation,

$$\frac{\partial n}{\partial t} = D\,\frac{\partial^2 n}{\partial x^2}\,,$$

with respect to x gives

$$\frac{d\tilde{n}(k,t)}{dt} = -Dk^2\tilde{n}(k,t)\,,$$

where $\tilde{n}(k,t)$ is the spatial Fourier transform of $n(x,t)$. We have written the derivative as a *total* derivative because k is just a parameter in this equation (there are no derivatives w.r.t. k) and so is effectively a constant. Hence, by using Fourier Transforms we have reduced the *partial* differential equation to an *ordinary* differential equation (which can easily be solved). The solution is

$$\tilde{n}(k,t) = \tilde{n}(k,0)\exp(-Dk^2t)\,. \tag{1}$$

Now

$$\tilde{n}(k,0) = \int_{-\infty}^{\infty} n(x,0)\,e^{ikx}\,dx = Q\,,$$

where the last equality follows because $n(x,0) = Q\,\delta(x)$, so Eq. (1) gives

$$\tilde{n}(k,t) = Q\exp(-Dk^2t)\,.$$

The inverse Fourier transform gives

$$n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{n}(k,t)\,e^{-ikx}\,dk = Q\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-Dk^2t)e^{-ikx}\,dk = \boxed{Q\,\frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}\,,}$$

where the last equality uses the result of doing a Gaussian integral by "completing the square", which we has been discussed in earlier classes. (See me if you have difficulty with this.)

Note: We see that

$$\int_{-\infty}^{\infty} x\,n(x,t)\,dx = 0$$

since the integrand is an odd function of x, and

$$\int_{-\infty}^{\infty} x^2\,n(x,t)\,dx = 2Dt\,,$$

which follows from standard properties of Gaussian integrals, which you should be getting familiar with by now. Hence the RMS value of x, $\langle x^2(t)\rangle^{1/2}$, which is the size of a typical displacement at time t, varies as $t^{1/2}$. This is a characteristic feature of diffusion.

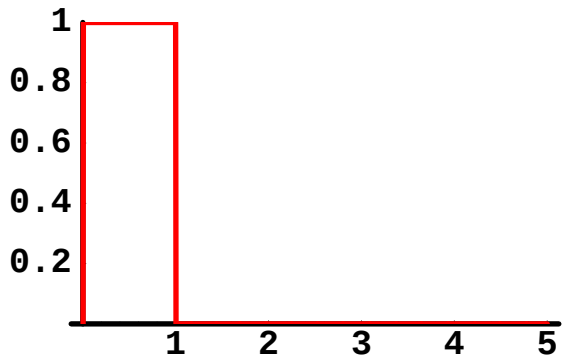
3. Denoting the sine Fourier transform of $u(x,t)$ by $g_s(k,t)$, we have

$$\frac{dg_s(k,t)}{dt} = -Dk^2g_s(k,t)\,,$$

where we have used the result (derived in class) that the sine Fourier transform of $\partial^2u(x,t)/\partial x^2$ is $-k^2g_s(k,t)$, provided $u(0,t) = 0$ (which is the case here). The solution is

$$g_s(k,t) = g_s(k,0)\exp(-Dk^2t)\,.$$

At $t = 0$ we have $u(x,0) = T_0(0 < x < 1), u(x,0) = 0(x > 1)$. (We are only interested in the range for $x > 0$). The figure below plots $u(x,0)/T_0$ against x.



We therefore have

$$g_s(k,0) = \int_0^\infty u(x,0)\,\sin(kx)\,dx = T_0 \int_0^1 \sin(kx)\,dx = T_0\,\frac{1-\cos k}{k}\,.$$

The inverse sine Fourier transform gives

$$u(x,t) = \frac{2}{\pi}\int_0^\infty g_s(k,t)\sin(kx)\,dx = T_0\,\frac{2}{\pi}\int_0^\infty \left[\frac{1-\cos k}{k}\right]\sin(kx)\exp(-Dk^2t)\,dk. \tag{2}$$

This integral is not trivial. To evaluate it we first differentiate with respect to x under the integral sign which removes the nasty factor of k in the denominator:

$$\frac{\partial u(x,t)}{\partial x} = T_0\,\frac{2}{\pi}\int_0^\infty [1-\cos k]\,\cos(kx)\exp(-Dk^2t)\,dk. \tag{3}$$

Consider the integral involving the "1" in the rectangular brackets.

$$\begin{aligned}\int_0^\infty \cos(kx)\exp(-Dk^2t)\,dk &= \frac{1}{2}\int_0^\infty \left[e^{ikx}+e^{-ikx}\right]\exp(-Dk^2t)\,dk = \frac{1}{2}\int_{-\infty}^\infty e^{ikx}\,\exp(-Dk^2t)\,dk \\ &= \frac{\exp[-x^2/(4Dt)]}{2}\int_{-\infty}^\infty \exp(-Dq^2t)\,dq = \sqrt{\frac{\pi}{4Dt}}\exp[-x^2/(4Dt)] \end{aligned} \tag{4}$$

where $q = k - ix/(2Dt)$, and we deformed the contour resulting from the change of variables back to the real axis. For the "$\cos k$" term in Eq. (3) we write

$$\cos k\,\cos(kx) = \frac{1}{2}\,\{\cos[(x+1)k] + \cos[(x-1])k\}\,. \tag{5}$$

Substituting into Eq. (3) we have two more integrals of the same type as in Eq. (4) and so we have

$$\frac{\partial u(x,t)}{\partial x} = T_0\,\frac{2}{\sqrt{4\pi Dt}}\left[\exp[-x^2/(4Dt)] - \frac{1}{2}\exp[-(x-1)^2/(4Dt)] - \frac{1}{2}\exp[-(x+1)^2/(4Dt)]\right]. \tag{6}$$

Integrating with respect, noting that $u(0,t)=0$ gives

$$\begin{aligned} u(x,t) &= T_0\,\frac{2}{\sqrt{4\pi Dt}}\int_0^x \left[\exp[-x'^2/(4Dt)] - \frac{1}{2}\exp[-(x'-1)^2/(4Dt)] - \frac{1}{2}\exp[-(x'+1)^2/(4Dt)]\right]dx'\,, \\ &= T_0\,\frac{2}{\sqrt{\pi}}\left[\int_0^{x/\sqrt{4Dt}} e^{-u^2}\,du + \frac{1}{2}\int_{1/\sqrt{4Dt}}^{(1-x)/\sqrt{4Dt}} e^{-u^2}\,du - \frac{1}{2}\int_{1/\sqrt{4Dt}}^{(1+x)/\sqrt{4Dt}} e^{-u^2}\,du\right], \end{aligned} \tag{7}$$

where we defined $u = (1 - x')/\sqrt{4Dt}$ in the second factor and $u = (1 + x')/\sqrt{4Dt}$ in the third factor. The lower limit in the second and third factors can be replaced by zero, since the extra added pieces cancel (note the relative minus sign of these two terms). We therefore have

$$u(x,t) = T_0 \frac{2}{\sqrt{\pi}} \left[\int_0^{x/\sqrt{4Dt}} e^{-u^2}\, du + \frac{1}{2} \int_0^{(1-x)/\sqrt{4Dt}} e^{-u^2}\, du - \frac{1}{2} \int_0^{(1+x)/\sqrt{4Dt}} e^{-u^2}\, du \right] . \tag{8}$$

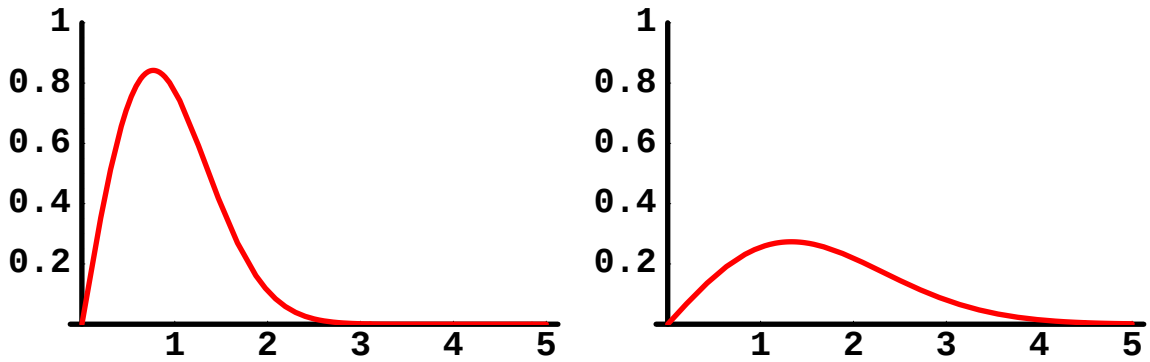
Using the definition of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du\,, \tag{9}$$

we obtain the final result

$$\boxed{u(x,t) = T_0 \left[\mathrm{erf}\left(\frac{x}{\sqrt{4Dt}}\right) + \frac{1}{2}\,\mathrm{erf}\left(\frac{1-x}{\sqrt{4Dt}}\right) - \frac{1}{2}\,\mathrm{erf}\left(\frac{1+x}{\sqrt{4Dt}}\right) \right] .} \tag{10}$$

The solution, $u(x,t)/T_0$, is plotted against x below for $Dt = 0.2$ (left) and $Dt = 0.8$ (right).



Note that the sharp peak which existed at $t = 0$ has become rounded out. Furthermore, you see that the peak becomes broader and less high with increasing time. This is typical diffusive behavior.

Note: I emphasize that by using the sine Fourier transform, we ensure that $u(0,t) = 0$ as required.

4. Fourier transforming

$$-D\frac{d^2 n(x)}{dx^2} + \kappa^2 D n(x) = Q\delta(x)\,,$$

with respect to x gives

$$Dk^2 g(k) + D\kappa^2 g(k) = Q\,, \tag{11}$$

where

$$g(k) = \int_{-\infty}^{\infty} n(x) e^{ikx}\, dx\,.$$

Equation (11) gives

$$g(k) = \frac{Q}{D(k^2 + \kappa^2)}\,.$$

The inverse transform is

$$n(x) = \frac{Q}{2\pi D} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{k^2 + \kappa^2}\, dk\,.$$

The evaluation of an almost identical integral is done for Qu. 1. One completes the contour in the lower half plane if $x > 0$ and in the upper half plane if $x < 0$. In each case one picks up a contribution from a simple pole and find

$$\boxed{n(x) = \frac{Q}{2\kappa D} e^{-\kappa|x|}\,.}$$

In other words, the concentration of neutrons decays exponentially away from the source at the origin. The reason for the decay is absorption, which is parametrized by the variable κ.

5. (a) There is one way of getting 12, $(6,6)$, two ways of getting 11, $(6,5),(5,6)$, and three of getting 10, $(6,4),(5,5),(4,6)$. Hence there are $1+2+3=6$ ways of getting a total greater than 9. There are $6^2 = 36$ ways in total of throwing two dice, so the probability of getting a total greater than 9 is $\boxed{6/36 = 1/6.}$

   (b) Proceeding as before the number of ways, n_{tot}, of getting different totals is shown in the table:

Total	n_{tot}
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1

   i.e. $\boxed{\text{the most likely result is 7}}$ which has the following 6 ways of achieving it: $(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)$. The probability of getting a 7 is $6/36$ $\boxed{= 1/6.}$

6. (a) The probability of a 3 is $3/21 = 1/7$. To obtain the probability of two events which are statistically independent we multiply the probabilities of the individual events, so the probability of two successive threes is $1/7^2$ $\boxed{= 1/49.}$

   (b) The probability of a 4 is $4/21$ and the probability of not throwing a 4 is $1 - 4/21 = 17/21$. Hence the probability of a 4 followed by not a 4 is $(4/21)(17/21)$ $\boxed{= 68/441.}$

   (c) The different ways of getting 10 or greater and their probabilities are

Numbers on dice	Total	Probability
$(4,6)$	10	$(4/21)(6/21) = 24/441$
$(5,5)$	10	$(5/21)^2 = 25/441$
$(6,4)$	10	$(6/21)(4/21) = 24/441$
$(5,6)$	11	$(5/21)(6/21) = 30/441$
$(6,5)$	11	$(6/21)(5/21) = 30/441$
$(6,6)$	12	$(6/21)^2 = 36/441$

The total probability of getting 10 or greater is $(24 + 25 + 24 + 30 + 30 + 36)/441 = 169/441$. Hence, the probability of two fives *given that the sum is* 10 *or greater* is $(25/441) \div (169/441)$ $\boxed{= 25/169.}$

7. Pick one out of the n people. Then choose a second person. The probability that he has a different birthday from the first is $1 - 1/365$. Assume that this is the case and then consider a third person. The probability that he has a different birthday from the first two is $1 - 2/365$. The probability that *all three* have different birthday is then the product of two factors: (i) the probability that the first two are different, and (ii) the probability that the third is different from both of the first two: i.e. $(1 - 1/365)(1 - 2/365)$. Continuing in this way the probability that all n people have different birthdays is given by a product of $n - 1$ factors as follows:

$$\boxed{p = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right)\cdots\left(1 - \frac{n-1}{365}\right).}$$

Hence

$$\ln p = \sum_{k=1}^{n-1} \ln\left(1 - \frac{k}{365}\right).$$

Assuming that $k \ll 365$ we can expand the log to first order, i.e. $\ln(1 + x) = x + \cdots$ and so

$$\ln p \simeq -\frac{1}{365}\sum_{k=1}^{n-1} k = -\frac{(n-1)n}{2 \times 365},$$

so

$$\boxed{p = \exp\left[-\frac{n(n-1)}{2 \times 365}\right],}$$

to a good approximation. The value of n where $p = 1/2$ is given by

$$\frac{n(n-1)}{730} = \ln 2,$$

which gives $n = 22.999$. Since n must, of course, be an integer, it follows that for $\boxed{n \geq 23}$ the probability of at least two of the people having the *same* birthday is greater than $1/2$.

8. The probability that the basketball player does *not* score with one throw is $1 - 3/4 = 1/4$. The probability that the player does not score in n throws is therefore $(1/4)^n$. We require this to be less that 0.01, i.e. $(1/4)^n < 0.01$, which is equivalent to $4^n > 100$. Since n is an integer the smallest value is $\boxed{n = 4.}$

9. The probability of a person having the disease is given by $P(d) = 10^{-4}$.

   The probability of a false negative, i.e. the probability that the person has the disease though the test is negative, is given in the question as $P(-|d) = 0.05$. Hence $P(+|d)$, the probability of testing positive given that the person has the disease, is given by $P(+|d) = 1 - P(-|d) = 0.95$.

   The probability that the person does not have the disease is given by $P(h) = 1 - P(d) = 0.9999$ ("h" for healthy). The probability of a false positive, i.e. the probability that the person is healthy but the test is positive, is given in the question as $P(+|h) = 0.01$. Consequently $P(-|h) = 1 - P(+|h) = 0.99$.

We are asked to determine the conditional probability that the person has the disease if he tests positive, i.e. $P(d|+)$. According to Bayes' theorem this is given by

$$\begin{aligned} P(d|+) &= \frac{P(+|d)\,P(d)}{P(+)} = \frac{P(+|d)\,P(d)}{P(+|d)\,P(d) + P(+|h)\,P(h)} \\ &= \frac{0.95 \times 10^{-4}}{(0.95 \times 10^{-4}) + (0.01 \times 0.9999)} \\ &= \frac{0.95}{0.95 + 99.99} \boxed{= 0.0094.} \end{aligned}$$

In other words, even if someone tests positive, the most likely situation is that he does not have the disease. However, the probability that he has the disease is much greater than the estimated probability before the test was done, which was 10^{-4}. In Bayesian analysis, the probability of having the disease before the test was done (based on any information up to that point) is called the "prior" probability (10^{-4} here), while the probability obtained by including additional information from the test is called the "posterior" probability (0.0094 here).