**PHYSICS 116C**
**Homework 10**

Due on class on Thursday December 5 (the last class). This homework will be graded. Solutions will be posted on the class web site after the class.

**Final Exam: The final exam will be on Tuesday December 10, 4:00–7:00 pm.**

I am hoping to have a second lecture room in which some of you will take the exam, so keep your eyes on email and the class website to **see where you should take the exam.**

The final will be cumulative but with a slight emphasis on the material since the midterm. The exam will be closed book, but you may bring in one sheet of notes that you have prepared yourself. Calculators, or other electronic devices, are not allowed.

1. A radioactive object has, *on average* 5 decays per second. What is the probability of detecting
   (a) 0
   (b) 1
   (c) 5
   (d) 20

   decays in a second.

2. A casino claims that there is a 50/50 chance of winning at a particular game. I play 100 times and lose 70 times.
   (a) Show that this corresponds to 4 times the standard deviation if the probability of winning is really 50%.
   (b) We showed in class that the distribution for this problem (the binomial distribution) goes over to a Gaussian distribution for a large number of trials. Since 100 is quite a large number we assume that this is a good approximation here. Assuming, then a Gaussian distribution show that the probability of a $\pm 4\sigma$ deviation, or greater, is $\mathrm{erfc}(4/\sqrt{2})$, where erfc is the complementary error function discussed earlier in the course.
   (c) You are given that $\mathrm{erfc}(4/\sqrt{2}) = 0.00006334$. Do you think that the casino was being honest in claiming a 50/50 chance of winning?

3. Consider the Lorentzian probability distribution

$$P(x) = \frac{1}{\pi}\frac{1}{1+x^2}\,.$$

   Choosing $N$ random variables, $x_1, x_2, \cdots, x_N$, we form the sum,

$$X = \sum_{i=1}^{N} x_i\,.$$

   Using Fourier transforms (called characteristic functions in this context), determine $P_N(X)$, the probability distribution of the sum. Show that distribution of the sample mean $\overline{x} = X/N$ is *identical* to the distribution of a single variable $P(x)$.
   *Note:* For a Gaussian distribution (or any other distribution with a finite variance, $\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$, for which the "central limit theorem" holds) the distribution of the sample mean is *narrower* than the distribution of a single data point (by a factor of order $1/\sqrt{N}$). Hence, in these cases, for

large $N$, the sample mean is very close to the exact mean of the distribution. However, here, the variance is not finite, the central limit theorem does not hold, and the distribution of the sample mean is just as broad as the distribution of $x$. The reason is that, since the distribution only falls off slowly at large $x$, there is a significant probability of getting in the sample some values which are much larger, in magnitude, than the typical value, and these dominate the mean.

4. Consider the exponential distribution

$$P(x) = \begin{cases} \frac{1}{x_0} \exp(-x/x_0), & (x \geq 0), \\ 0, & (x < 0). \end{cases}$$

   (a) Show that the mean $\mu$ and standard deviation $\sigma$ are both equal to $x_0$.

   (b) Determine the distribution of a sum of $N$ independent random variables each of which has this distribution, i.e. determine $P_N(X)$ where $X = \sum_{i=1}^{N} x_i$.
   *Recall:* The procedure is (i) determine $g(k)$, the Fourier transform of $P(x)$, and (ii) compute the inverse Fourier transform of $g^N(k)$. For part (ii) you will need to do a contour integral and remember how to compute the residue at an $N$-th order pole.
   *Answer:*

$$P_N(X) = \begin{cases} \frac{1}{(N-1)!} \frac{X^{N-1}}{x_0^N} \exp(-X/x_0) & (X \geq 0), \\ 0, & (X < 0). \end{cases}$$

   (c) Show that the mean and variance of $P_N(X)$ are given by

$$\mu_X = N\mu, \tag{1}$$
$$\sigma_X^2 = N\sigma^2. \tag{2}$$

   *Hint:* It is useful to recall the definition of the $\Gamma$ function.
   *Note:* Equations (1) and (2) are true *in general* for any distribution (in which the mean and standard deviation are finite).
   In addition, *for large* $N$, the central limit theorem tells us not only the mean and variance of the distribution but also the form of the distribution itself, namely that it is a Gaussian. You may wish to show this for the present problem by expanding $X$ away from its most probable value (i.e. the value where $P_N(X)$ is maximum).

5. I have just two measurements of a quantity which is subject to random noise: 1 and 3. What is my best estimate for the mean, and the error bar in this estimate?

6. I carry out the following series of measurements for a particular quantity, $x$, which is subject to random noise:

$$1.1,\ 0.9,\ 0.95,\ 1.05,\ 1.0\,.$$

   What is my best estimate for $\langle x \rangle$ and what is the error bar on this estimate?