PHYSICS 116A

Final Examination, 2013
Thursday March 21, 12:00 – 3:00 pm

LOCATION

- If your name is in the range BARKER-MOIR, you must take the exam in the usual lecture room, Thimann 1.
- If your name is in the range MONTERO-YEUNG, you must take the exam in Physical Sciences 130. (You enter this room from OUTSIDE the building).

CLOSED BOOK.
You are allowed to use one sheet of hand written notes on which you can write anything you wish.
No calculators or other electronic devices are permitted.

TO GET CREDIT YOU MUST SHOW YOUR WORKING.

Write your answers on separate sheets of paper, not on the question sheet.
PLEASE ANSWER EACH QUESTION ON A SEPARATE SHEET.

1. **[14 points]**

   (a) Express in the form $x + iy$

   i. $\cosh(2\pi i)$
   ii. $\tan(i)$
   iii. 2^i

   (b) Show that
   $$\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1+iz}{1-iz}\right).$$
   and hence that $\ln[\,(1+i)/(1-i)\,] = i\pi/2$.

2. **[14 points]**

   (a) Consider the series
   $$1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots .$$
   Use the ratio test to determine the radius of convergence.

   (b) Consider the series
   $$-x + \frac{x^2}{2^2} - \frac{x^3}{3^2} + \frac{x^4}{4^2} - \cdots .$$
   Use tests *other than the ratio test* (which is not useful here) to determine whether the series converges at $x = 1$ and $x = -1$.
   Note: For each case, you must explain which test you are using.

3. **[12 points]**

   (a) Evaluate
   $$\frac{3\,\Gamma(5/4)}{2\,\Gamma(1/4)}\,.$$

   (b) Evaluate
   $$\int_0^\infty x^{1/2}\,e^{-x^3}\,dx$$
   in terms of a gamma function.

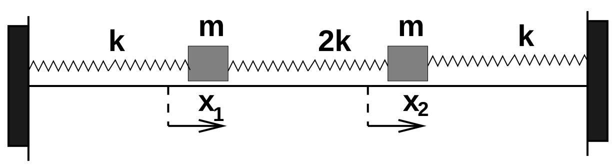
4. **[14 points]**
   Solve the following set of equations using Gaussian elimination:
   $$\begin{aligned} 2x + 5y + z &= 5\,, \\ x + 4y + 2z &= 1\,, \\ 4x + 10y - z &= 1\,. \end{aligned}$$
   Check your answer by substituting the values for x, y and z back into the equations.

5. **[14 points]**

   (a) If A and B are square matrices does $(A+B)^2$ always equal $A^2 + 2AB + B^2$? Does $(AB)^2$ always equal $A^2\,B^2$?

   (b) An $n \times n$ matrix is diagonalized by the similarity transformation $D = C^{-1}MC$, where D is a diagonal matrix. Show that the determinant of M is equal to the product of the eigenvalues, *i.e.*
   $$\det(M) = \prod_{\alpha=1}^{n} \lambda_\alpha\,,$$
   where λ_α is an eigenvalue.
   Hint: Consider $\det(D)$.

6. **[18 points]**
   Consider two equal masses, m, connected by springs with spring constants k and $2k$ as shown, attached end to end with the ends of the left hand and right hand spring fixed. They slide on a horizontal frictionless surface. Let x_1 and x_2 be the positions of the masses relative to their position when the springs are not stretched.



   (a) Show that the energy is given by
   $$E = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \tfrac{1}{2}k(3x_1^2 + 3x_2^2 - 4x_1x_2).$$

   (b) Find the frequencies of the normal modes of oscillation (i.e. the oscillations which occur at a single frequency).

(c) Describe qualitatively the relative motion of the two masses for each of the normal modes.

7. **[14 points]**

(a) You are given that

$$
\begin{aligned}
\epsilon_{123} = \epsilon_{231} = \epsilon_{312} &= 1\,, \\
\epsilon_{132} = \epsilon_{213} = \epsilon_{321} &= -1\,, \\
\epsilon_{ijk} &= 0 \quad \text{if any two indices are equal,} \\
\delta_{ij} &= 1 \quad \text{if } i = j\,, \\
\delta_{ij} &= 0 \quad \text{if } i \neq j\,,
\end{aligned}
$$

(where i and j run over values 1, 2, and 3). Show that

i. $\delta_{ii} = 3$

ii. $\delta_{ij}\epsilon_{ijk} = 0$

iii. $\epsilon_{ijk}\epsilon_{ijk} = 6$,

(summation over repeated indices implied.)

(b) Consider *two*-dimensional space. Show that

$$\epsilon_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is an *isotropic* second rank tensor, *i.e.* it is a tensor which is the same in all rotated frames of reference.

Hint: If you recall the explicit form for a rotation matrix in two dimensions you may use that. However, all you really need is one general property of a rotation matrix. Which property is that?