

## Physics 116A

### Inverting a matrix by Gauss-Jordan elimination

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In [http://young.physics.ucsc.edu/116A/gauss\\_elim.pdf](http://young.physics.ucsc.edu/116A/gauss_elim.pdf) we discussed how to solve linear equations by Gaussian elimination in which we transform the “augmented matrix” to “row echelon form”. In Appendix C of that reference we showed that it is also possible to solve the equations by further reducing the augmented matrix to “reduced row echelon form”, a procedure known as Gauss-Jordan elimination. We pointed out there that if the matrix of coefficients is square, then, provided its determinant is non-zero, its reduced echelon form is the identity matrix.

Here we show how to determine a matrix inverse (of course this is only possible for a square matrix with non-zero determinant) using Gauss-Jordan elimination. This is much more efficient than using the textbook formula: the transpose of the matrix of cofactors divided by the determinant. With Gauss-Jordan reduction, the number of operations to invert an  $n \times n$  matrix is of order  $n^3$ . With the textbook formula, even if one evaluates the cofactors (which are determinants of order  $n-1$ ) by an efficient algorithm, each one requires of order  $(n-1)^3$  operations. However there are  $n^2$  for them so the operation count is order  $n^5$ , which is much bigger than  $n^3$  for large  $n$ . Even worse is to evaluate the cofactors by “expanding” along a row or column, repeating for successively smaller cofactors, until one gets down to  $2 \times 2$  matrices which are evaluated explicitly. As discussed in the handout on determinants, <http://young.physics.ucsc.edu/116A/determinants.pdf>, the number of operations grows like  $n!$  which increases so fast that it would take the age of the universe to find the determinant of a  $25 \times 25$  matrix.

The transformations used in row reduction are:

- Interchange two rows.
- Multiply a row by a constant.
- Add a multiple of one row to another.

Each of these operations can be regarded as multiplying the matrix on the left by another matrix which does the transformation. As an example, suppose we have a  $3 \times 3$  matrix and we want to

add  $k$  times row 2 to row 1. This is accomplished by multiplying on the left by

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

since

$$\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \quad (2)$$

We will perform a sequence of row transformations to transform the matrix  $A$  to the identity matrix and *simultaneously perform the same sequence of operations on the identity matrix*  $I$ . Suppose these transformations are carried out by multiplying by matrix  $X_1$  (first), then  $X_2, X_3$  etc. and finally  $X_p$ . Then  $X_p X_{p-1} \cdots X_2 X_1 A = I$  (note the order of operations). Hence  $X_p X_{p-1} \cdots X_2 X_1 = A^{-1}$  since  $A^{-1} A = I$ . Applying the same transformations to the identity matrix we get  $X_p X_{p-1} \cdots X_2 X_1 I = A^{-1} I = A^{-1}$ , i.e. the inverse of matrix  $A$ . In other words

$$\boxed{\text{if } X_p X_{p-1} \cdots X_2 X_1 A = I \quad \text{then} \quad X_p X_{p-1} \cdots X_2 X_1 I = A^{-1}.} \quad (3)$$

Lets apply this Gauss-Jordan elimination to a particular example.

$$\boxed{A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.} \quad (4)$$

We start by writing  $A$  and the identity matrix side by side and then performing row operations. In each case where we add a multiple of one row to another, the pivot element is shown by putting a 1 around it and coloring it green:

$$\begin{pmatrix} \boxed{1} & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R^{(3)} \rightarrow R^{(3)} - 2R^{(1)}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \xrightarrow{R^{(3)} \rightarrow R^{(3)} - 4R^{(2)}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & 1 \end{pmatrix}$$

$$\begin{array}{ccc}
\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & 1 \end{pmatrix} & \xrightarrow{R^{(3)} \rightarrow -\frac{1}{4}R^{(3)}} & \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix} \\
\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \boxed{1} \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix} & \xrightarrow{R^{(2)} \rightarrow R^{(2)} - R^{(3)}} & \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix} \\
\begin{pmatrix} 1 & -1 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix} & \xrightarrow{R^{(1)} \rightarrow R^{(1)} + R^{(2)}} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix}. \quad (5)
\end{array}$$

The left hand matrix, which started as  $A$ , has been transformed into the identity matrix, so the right hand matrix, which started as the identity matrix, has been transformed into  $A^{-1}$ . Thus

$$\boxed{A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix}}. \quad (6)$$

It is easy to verify by multiplying out the terms that

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{2} & 1 & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (7)$$

as required.

While this evaluation of the matrix inverse might seem rather laborious, we note that much of the work is copying unchanged rows from line to line, which is trivial, and does not have to be done at all by the computer code in a computer evaluation. The crucial point is that the operation count only increases slowly with  $n$ , like  $n^3$ , rather than  $n!$  if one uses Cramer's rule and evaluates the determinants the textbook way by expanding along a row or column.