Homework 7, Solutions

- 1 (a)

\[
\text{sphericalBessel}[n_, x_] := \text{Sqrt}[\text{Pi}/2]/\text{Sqrt}[x] \text{BesselJ}[n+1/2, x] // \text{Apart}
\]

\[
\text{sphericalBessely}[n_, x_] := \text{Sqrt}[\text{Pi}/2]/\text{Sqrt}[x] \text{BesselY}[n+1/2, x] // \text{Apart}
\]

Note: I wrote the factor involving the square root of \(x\) as \(1/\sqrt{x}\) rather than \(\sqrt{1/x}\) to avoid having square roots that \textit{Mathematica} won’t cancel. (Alternatively I could have used \textit{Simplify} with the condition that \(x > 0\).) \textit{Apart} presents the result in a neater way.

- 1 (b)

\[
\{\text{sphericalBessel}[0, x], \text{sphericalBessely}[0, x]\}
\]

\[
\left\{\frac{\text{Sin}[x]}{x}, -\frac{\text{Cos}[x]}{x}\right\}
\]

\[
\{\text{sphericalBessel}[1, x], \text{sphericalBessely}[1, x]\}
\]

\[
\left\{-\frac{\text{Cos}[x]}{x} + \frac{\text{Sin}[x]}{x^2}, -\frac{\text{Cos}[x]}{x^2} - \frac{\text{Sin}[x]}{x}\right\}
\]

\[
\text{sphericalBessel}[10, x]
\]

\[
\frac{55 \left(11904165 - 1670760 x^2 + 51597 x^4 - 468 x^6 + x^8\right) \text{Cos}[x]}{x^{10}}
\]

\[
\left(-654729075 + 310134825 x^2 - 18918900 x^4 + 315315 x^6 - 1485 x^8 + x^{10}\right) \text{Sin}[x]
\]

\[
\text{sphericalBessely}[10, x]
\]

\[
\left(-654729075 + 310134825 x^2 - 18918900 x^4 + 315315 x^6 - 1485 x^8 + x^{10}\right) \text{Cos}[x]
\]

\[
\frac{55 \left(11904165 - 1670760 x^2 + 51597 x^4 - 468 x^6 + x^8\right) \text{Sin}[x]}{x^{10}}
\]

- 1 (c)

\[
\text{Series}[\text{sphericalBessel}[10, x], \{x, 0, 15\}]
\]

\[
\frac{x^{10}}{13749310575} - \frac{x^{12}}{632468286450} + \frac{x^{14}}{63246828645000} + O[x]^{16}
\]

\[
\text{Series}[\text{sphericalBessely}[10, x], \{x, 0, -6\}]
\]

\[
-\frac{654729075}{x^{11}} - \frac{3459425}{2 x^3} - \frac{2027025}{8 x^7} + \frac{1}{O[x]^5}
\]
1 (d)

Plot[{sphericalBesselJ[1, x], sphericalBesselY[1, x]}, {x, 0, 10}, PlotRange -> {-1.5, 0.5}]

2 (a)

ser = Series[Tanh[x], {x, 0, 20}]

\[
\frac{x^3}{3} + 2 x^5 + \frac{17 x^7}{15} + \frac{62 x^9}{315} + \frac{1382 x^{11}}{2835} + \frac{21844 x^{13}}{6081075} - \frac{929569 x^{15}}{638512875} + \frac{6404582 x^{17}}{10854718875} - \frac{443861162 x^{19}}{1856156927625} + O[x]^{21}
\]

2 (b)

InverseSeries[ser]

\[
\frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \frac{x^{13}}{13} + \frac{x^{15}}{15} + \frac{x^{17}}{17} + \frac{x^{19}}{19} + O[x]^{21}
\]

Note: \( \tanh^{-1}(x) = (1/2) \ln[(1+x)/(1-x)] \), and we indeed recover the series for this function.

2 (a)

At low \( T \) the upper limit of the integral can be replaced by \( \infty \). The integral is

\[
\text{Integrate}[x^4 \exp[x] / (\exp[x] - 1)^2, \{x, 0, \infty\}]
\]

\[
\frac{4 \pi^2}{15}
\]

and multiplying by 9 gives the answer.

\[
N[12 \pi^4 / 5.]
\]

233.782
3 (b)
At high-$T$ the upper limit of the integral is small so we can expand the integrand, which becomes $x^2$. The integral is then trivial and leads to the answer given in the question.

3 (c)

Plot $B$:

```math
Plot\left[\int t^3 \text{NIntegrate}\left[\frac{x^4 e^x}{(e^x - 1)^2}, \{x, 0, \frac{1}{t}\}\right], \{t, 0.01, 1.3\},
\right.
\text{AxesLabel} \rightarrow \{"T/\theta_D", "C/Nk_B"\}, \text{PlotRange} \rightarrow \{0, 3.2\}, \text{PlotStyle} \rightarrow
\{\{\text{Hue}[0], \text{AbsoluteThickness}[3]\}, \{\text{Hue}[0.7], \text{AbsoluteThickness}[3], \text{Dashing}[\{0.01, 0.03\}]\}\}
```

![Graph of C/Nk_B vs. T/\theta_D](image)

4(a)

```math
Plot\left[\{m, \text{Tanh}\left[\frac{m}{T}\right] / \Rightarrow 1.5\}, \{m, -2, 2\}, \text{PlotRange} \rightarrow \{-1.1, 1.1\}\right]
```

![Graph of m vs. T](image)

The only solution is for $m = 0$. This will be the case for all $T > 1$ because $\tanh(x) = x$ for small $x$. 
4(b)

\[ \text{Plot}\left[ \left\{ m, \text{Tanh}\left(\frac{m}{T}\right) / . T \to 0.5 \right\}, \left\{ m, -2, 2 \right\}, \text{PlotRange} \to \{-1.1, 1.1\} \right] \]

We see that there are now two additional solutions with equal and opposite magnetization.

4(c)

\[ \text{Plot}\left[ m / . \text{FindRoot}\left[ m = \text{Tanh}\left(\frac{m}{T}\right), \left\{ m, 0.5 \right\} \right], \left\{ T, 0.02, 1.2 \right\}, \right. \]
\[ \left. \text{AxesLabel} \to \{"T/T_c", "m"\}, \text{PlotStyle} \to \{\text{Hue}[0], \text{AbsoluteThickness}[5]\}] \]

4(d)

Since \( \tanh(x) = x - x^3/3 + \ldots \), we have

\[ m \left(1 - \frac{J}{T}\right) = \frac{1}{3} \left(\frac{J}{T}\right)^3 m^3 \]

for \( m \) small. Either \( m = 0 \) or \( m^2 = 3 (T_c - T) / T_c \) (to lowest order in \( T_c - T \)) with \( T_c = J \).
4(e)

\[
\text{Plot}\left[\left\{m^2 / . \text{FindRoot}\left[m = \text{Tanh}\left[\frac{m}{T}\right], \{m, 0.5\}\right], \{T, 0.02, 1.2\}\right\}, \text{AxesLabel} \to \{"T/T_e", "m^2"\}, \text{PlotRange} \to \{(0, 1.2), \{0, 1.1\}\}, \text{PlotStyle} \to \{(\text{Hue}[0], \text{AbsoluteThickness}[3]), (\text{Hue}[0.7], \text{AbsoluteThickness}[3], \text{Dashing}[[0.01, 0.03]])\}\right]
\]

The dashed line is the expected result for \( T \) close to \( T_e \). It agrees well with the full numerical results shown in the solid line.

5(a)

This solution follows closely the handout on "The range of a baseball including air resistance" which should be consulted for further details.

\text{Clear["Global`*"]}
\text{g} = 9.81;

We determine the time, \( t_{\text{final}} \), when the particle hits the ground as a function of \( \theta \).

\( t_{\text{final}}[\theta_] := (\text{sol} = \text{NSolve[}
\{x''[t] = -k x'[t] \text{Sqrt}[y'[t]^2 + x'[t]^2], y''[t] = -k y'[t] \text{Sqrt}[y'[t]^2 + x'[t]^2] - g,
\quad x[0] = 0, y[0] = 0, x'[0] = v \cos[\theta], y'[0] = v \sin[\theta], \{x, y\}, \{t, 0, 20\}\};
\text{yy}_{[\_]} = y[t] / . \text{sol}[[1]]; \text{xx}_{[\_]} = x[t] / . \text{sol}[[1]];
\text{t} /. \text{FindRoot}[\text{yy}[t], \{t, 0.1, 20\}, \text{MaxIterations} \to 50])
\)

Next we determine the horizontal distance traveled, \( x_{\text{final}} \), when the particle hits the ground (i.e. the range) as a function of \( \theta \).

\( x_{\text{final}}[\theta_?\text{NumericQ}] := \text{xx}[t_{\text{final}}[\theta]]\)

We test this function for \( v = 20 \), \( k = 0 \).

\( v = 20; \quad k = 0; \quad \text{Table}[x_{\text{final}}[\theta], \{\theta, 0.1, 1.0, 0.1\}]\)
\{8.10069, 15.8784, 23.0231, 29.25, 34.3107, 38.0036, 40.1814, 40.7573, 39.7084, 37.0763\}

by checking that it gives \( v^2 \sin 2\theta / g \),

\( \text{Table}[v^2 \sin[2\theta] / g, \{\theta, 0.1, 1.0, 0.1\}]\)
\{8.10069, 15.8784, 23.0231, 29.25, 34.3107, 38.0036, 40.1814, 40.7573, 39.7084, 37.0763\}
and indeed it does. We next optimize the range with respect to $\theta$. The maximum range, $x_{\text{max}}$ will be evaluated as a function of $v$. We will also compute the optimal angle $\theta_{\text{max}}$. Note how we indirectly pass the value of $v$ in the argument of $x_{\text{max}}$ and $\theta_{\text{max}}$ to the function $x_{\text{final}}[\theta]$. 

\[
x_{\text{max}}[v] := (v = vv; \text{FindMinimum}[-x_{\text{final}}[\theta], \{\theta, 0.1, 1.3\}][[1]])
\]

\[
\theta_{\text{max}}[v] := (v = vv; \theta \rightarrow \text{FindMinimum}[-x_{\text{final}}[\theta], \{\theta, 0.1, 1.3\}][2])
\]

We verify that with no air resistance, $k=0$, we correctly obtain $x_{\text{max}} = \frac{v^2}{g}$, $\theta_{\text{max}} = \frac{\pi}{4}$:

\[
x_{\text{max}}[10]
\]

10.1937

\[
\theta_{\text{max}}[10]
\]

0.785398

From now on, we set $k = 0.004$, as stated in the question.

\[
k = 0.004;
\]

First we consider $x_{\text{max}}$ for $v = 50$, with and without friction.

\[
v = 50;
\]

\[
\{x_{\text{max}}[v], \frac{v^2}{g}\}
\]

\[
\{150.21, 254.842\}
\]

Hence without friction the maximum range for initial speed $v = 50 \text{ m/sec}$ would be 255 meters while with a friction coefficient of $k=0.004$ it is reduced to 150 meters.

- 5(b)

Next we plot the results. The dashed line in the plots gives the result for no friction (i.e. $k=0$), $x_{\text{max}} = \frac{v^2}{g}$.

First of all we show that the effects of friction are small for small $v$. 

Next we show that friction reduces the range considerably at large $v$.

$$f(x) := 4 \lambda x (1 - x)$$

(i) $\lambda = 0.2$

$$\lambda = 0.2; \text{FixedPoint}[f, 0.7, \text{SameTest} \rightarrow (\text{Abs}[#1 - #2] < 10^{-11} & ) ]$$

$3.49271 \times 10^{-11}$

i.e fixed point (FP) at zero $x$ for $\lambda = 0.2$.

(ii) $\lambda = 0.6$
\[ \lambda = 0.6; \text{FixedPoint}[f, 0.7, \text{SameTest} \to (\text{Abs}[#1 - #2] < 10^{-11} \&)] \]

0.583333

i.e fixed point (FP) at non-zero \( x \) for \( \lambda = 0.6 \).

\[ f2[x_] := f[f[x]]; f4[x_] := f2[f2[x]] \]

(iii) \( \lambda = 0.8 \)

Either plot successive results for \( x \), which clearly shows a cycle of length 2, or show that FixedPoint for \( f \) does not converge, but FixedPoint for the second iterate, \( f2(x) \), does converge.

\[ \lambda = 0.8; \text{FixedPoint}[f2, 0.7, \text{SameTest} \to (\text{Abs}[#1 - #2] < 10^{-11} \&)] \]

0.799455

(iv) \( \lambda = 0.84 \)

Either plot successive results for \( x \), which clearly shows a cycle of length 2, or show that FixedPoint for \( f \) does not converge, but FixedPoint for \( f2(x) \), does converge.

\[ \lambda = 0.84; \text{FixedPoint}[f2, 0.7, \text{SameTest} \to (\text{Abs}[#1 - #2] < 10^{-11} \&)] \]

0.462375

(v) \( \lambda = 0.90 \)

FixedPoint does not converge for \( f, f2, f4 \) or any other iterate of \( f \) that one can try. The data looks chaotic

\[ \lambda = 0.9^\prime; \text{ListPlot}[\text{NestList}[f, 0.7, 80], 20], \text{PlotStyle} \to \{\text{Hue[0], PointSize[0.02]}\}] \]

so we conclude that there is chaos for \( \lambda = 0.90 \)

(vi) \( \lambda = 0.961 \)

Looking at the plot in the handout, this looks like a period-3 cycle. If we try to verify this using FixedPoint on the third iterate, \( f(x) \), we find that it does not converge. Actually 0.961 is just in the region where the period three has doubled to period 6, though this is difficult to see from the plots. One finds that FixedPoint with \( f6(x) \) does converge:

\[ \lambda = 0.961; f3[x_] := f[f[f[x]]]; f6[x_] := f3[f3[x]]; \text{FixedPoint}[f6, 0.7, \text{SameTest} \to (\text{Abs}[#1 - #2] < 10^{-11} \&)] \]

0.498668

(vii) \( \lambda = 0.97 \)

Chaos. Show a ListPlot
6(b)

Here we compute the Lyapunov exponent, $\lambda_L$, as discussed in class. First of all we define a function $\text{lya}[\ldots]$ which will determine $\lambda_L$ for a given value of $\lambda$. Note the use of the Mathematica construct $\text{Apply}[\text{Plus}, \text{list}]$ to sum the elements of a list, and also the function $\text{Length[list]}$, which gives the number of elements in the list.

```mathematica
fp[x_] := f'[x]
lya[la_, xinit_, n_, ndrop_] := (\lambda = la; xlist = Drop[ NestList[f, xinit, n], ndrop + 1]; Apply[Plus, Log[Abs[fp[xlist]]]] / Length[xlist])
```

Next we set up a list of the $\lambda$ values, and the corresponding Lyapunov exponents:

```mathematica
lvals = {0.2, 0.6, 0.8, 0.84, 0.9, 0.961, 0.97, 1}
lyapunovs = Table[lya[lvals[[n]], 0.7, 20000, 1000], {n, Length[lvals]}]
```

\[
\begin{align*}
\lambda &= 0.97; \quad \text{ListPlot}\left[\text{Drop[NestList}[f, 0.7, 80], 20], \text{PlotStyle}\rightarrow\{\text{Hue}[0], \text{PointSize}[0.02]\}\right] \\
\lambda &= 1; \quad \text{ListPlot}\left[\text{Drop[NestList}[f, 0.7, 80], 20], \text{PlotStyle}\rightarrow\{\text{Hue}[0], \text{PointSize}[0.02]\}\right]
\end{align*}
\]
Finally we display a pretty table of the results.

\[
\text{Print } \{"\lambda \quad \lambda_L \quad \}\text{; Print } \text{TableForm}[\text{Transpose}[[\text{lvals}, \text{lyapunovs}]]]
\]

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\lambda_L)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>-0.223144</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.916291</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.916291</td>
</tr>
<tr>
<td>0.84</td>
<td>-0.28141</td>
</tr>
<tr>
<td>0.9</td>
<td>0.181912</td>
</tr>
<tr>
<td>0.961</td>
<td>-0.280362</td>
</tr>
<tr>
<td>0.97</td>
<td>0.46721</td>
</tr>
<tr>
<td>1</td>
<td>0.693172</td>
</tr>
</tbody>
</table>

Note that \(\lambda_L\) is negative where the trajectory is a fixed point or limit cycle, i.e. \(\lambda = 0\) (FP at \(x=0\)), \(\lambda = 0.6\) (FP at non-zero \(x\)), \(\lambda = 0.8\) (period-2 cycle), \(\lambda = 0.84\) (period 2 cycle), and \(\lambda = 0.961\) (period 6 cycle). However \(\lambda_L\) is positive in the chaotic regime, i.e. for \(\lambda = 0.9, 0.97\) and 1.

For a FP at \(x = 0\), \(\lambda_L\) should equal \(\log(4 \lambda)\), (show this by evaluating the derivative at the fixed point) and the result for \(\lambda = 0.2\) agrees with this. For a FP at non-zero \(x\), \(\lambda_L\) should equal \(\log(|2 - 4 \lambda|)\) (again show this by evaluating the derivative at the FP), and the result for \(\lambda = 0.6\) agrees with this. For \(\lambda = 1\), the Lyapunov exponent appears, numerically, to equal \(\log(2)\). This result can also be shown analytically.

### 6(c)

By inspection of the figure in the handout, we try the region round 0.935.

\[\text{In}[2]:= \text{iterate}[m_, n_] := \text{Drop}[	ext{NestList}[f, 0.5, n], m]\]

\[\text{In}[3]:= \text{drawpt}[y_] := \text{Point}[[\lambda, y]]\]

\[\text{In}[4]:= \text{graph}[\text{m}, \text{max}, \text{min}, \text{mdrop}, \text{n}] := \text{Graphics}[[\text{PointSize}[0.001],\]
\[\text{Table}[\text{Map}[\text{drawpt, iterate[mdrop, n] }], \{\lambda, \text{min}, \text{max}, (\text{max} - \text{min})/\text{n})\}]]\]
We see that this does indeed contain a period-5 cycle.

- 6(d)

Now we blow up the cycle.
We see period doubling.

- 6(e)

Define \( \lambda_k \) to be the value of \( \lambda \) where the \( 2^{k-1} \)-cycle becomes unstable and goes into a period \( 2^k \) cycle. We have \( \lambda_0 = 1/4 \), and \( \lambda_1 = 3/4 \). By using FixedPoint on \( f_2 \) and \( f_4 \) we can determine, with fair precision when the length-2 cycle becomes unstable. Using bisec-tion (done by hand!) I deduce that \( \text{FixedPoint} \) for \( f_2 \) converges for \( \lambda = 0.8623 \)

\[
\lambda = 0.8623; \quad \text{FixedPoint}[f_2, 0.7, \text{SameTest} \rightarrow (\text{Abs}[#1 - #2] < 10^{-11} &)]
\]

0.440028

but not for \( \lambda = 0.8624 \), though \( f_4 \) does have a fixed point for \( \lambda = 0.8624 \);

\[
\lambda = 0.8624; \quad \text{FixedPoint}[f_4, 0.7, \text{SameTest} \rightarrow (\text{Abs}[#1 - #2] < 10^{-11} &)]
\]

0.442748

Hence I estimate \( \lambda_2 = 0.86235 \). Kinzel and Reents give the exact result as \( \left(1 + \sqrt{6}\right)/4 = 0.862372\ldots \). This is also shown using \textit{Mathematica} in the handout on the logistic map.

Next look for the FP of \( f_4(x) \) going instable. \( f_4 \) has a fixed point for \( \lambda = 0.886 \)

\[
\lambda = 0.886; \quad \text{FixedPoint}[f_4, 0.7, \text{SameTest} \rightarrow (\text{Abs}[#1 - #2] < 10^{-11} &)]
\]

0.363324

but not for \( \lambda = 0.8861 \), though \( f_8 \) does have a fixed point for \( \lambda = 0.8861 \);

\[
f_8[x_] := f_4[f_4[x]] // \text{Simplify}
\]
\[ \lambda = 0.8861; \text{FixedPoint}[f8, 0.7, \text{SameTest} \rightarrow (\text{Abs}[#1 - #2] < 10^{-11} \&)] \]

0.365049

Hence I estimate \( \lambda_3 = 0.88605 \)

Now we look where the fixed point of \( f_8 \) disappears. There is a fixed point for \( \lambda = 0.89105 \)

\[ \lambda = 0.89105; \text{FixedPoint}[f8, 0.7, \text{SameTest} \rightarrow (\text{Abs}[#1 - #2] < 10^{-11} \&)] \]

0.374712

but not for \( \lambda = 0.89110 \) so I estimate \( \lambda_4 = 0.89107 \).

Now we compute the Feigenbaum constant:

\[
\begin{align*}
\text{la}[0] &= 0.25; \, \text{la}[1] = 0.75; \, \text{la}[2] = 0.86235; \, \text{la}[3] = 0.88605; \, \text{la}[4] = 0.89107; \\
\text{Print}[k, " \, \delta_{k}:"];
\text{Do[Print}[k, " \, (\text{la}[k] - \text{la}[k-1])/(\text{la}[k+1] - \text{la}[k])], \{k, 1, 3\}] \\
0.004 & \, \delta_{0.004} \\
1 & \, 4.45438 \\
2 & \, 4.74051 \\
3 & \, 4.72112
\end{align*}
\]

These estimates compare fairly well with the correct value of 4.669...

This is sufficient for the homework. However, for the hell of it, we will now work a bit harder and get these three estimates more accurately and also obtain two additional estimates of \( \delta \)

\text{Clear["Global`"]}

It turns out to help \textit{Mathematica} by writing \( f(x) \) as follows:

\[
\begin{align*}
f[x_] &= 4 \lambda ((1/2)^2 - (x - 1/2)^2); \\
\text{Select}[\lambda \, /\, \text{NSolve}[\{f[x] = x, f'[x] = 1\}, \{x, \lambda\}], \text{Im}[#] = 0 \&] \\
\{0.25, 0.25\} \\
l[0] &= \%[[1]] \\
0.25 \\
\text{Select}[\lambda \, /\, \text{NSolve}[\{f[x] = x, f'[x] = -1\}, \{x, \lambda\}], \text{Im}[#] = 0 \&] \\
\{0.75, -0.25\} \\
l[1] &= \text{Select}[\%, 0 \times 0.25 < # < 1 \&][[1]] \\
0.75 \\
f2[x_] &= f[f[x]]; \\
\text{Select}[\lambda \, /\, \text{NSolve}[\{f2[x] = x, f2'[x] = -1\}, \{x, \lambda\}], \text{Im}[#] = 0 \&] \\
\{-0.362372, 0.862372, 0.862372, -0.362372\}
\end{align*}
\]
\[ l[2] = \text{Select}[\%, \ 0.75 < \% < 1 \&][1] \]
0.862372

\[ f4[x_] = f2[f2[x]]; \]
FindRoot[{f4[x] == x, f4'[x] == -1}, {x, 0.8}, {\lambda, 0.88}]  
{x \to 0.819785, \lambda \to 0.886023}

\[ l[3] = \lambda / . \% \]
0.886023

\[ f8[x_] = f4[f4[x]]; \]
FindRoot[{f8[x] == x, f8'[x] == -1}, {x, 0.891}, {\lambda, 0.89}]  
{x \to 0.890787, \lambda \to 0.891102}

\[ l[4] = \lambda / . \% \]
0.891102

\[ f16[x_] = f8[f8[x]]; \]
FindRoot[{f16[x] == x, f16'[x] == -1}, {x, 0.892}, {\lambda, 0.892}]  
{x \to 0.89214, \lambda \to 0.89219}

\[ l[5] = \lambda / . \% \]
0.89219

\[ f32[x_] = f16[f16[x]]; \]
FindRoot[{f32[x] == x, f32'[x] == -1}, {x, 0.8923}, {\lambda, 0.8923}]  
{x \to 0.892415, \lambda \to 0.892423}

\[ l[6] = \lambda / . \% \]
0.892423

Print[k, "", \delta_k];
Do[Print[n, "", (l[n] - l[n-1])/(l[n+1] - l[n])], {n, 1, 5}]
k \delta_k  
1 4.44949
2 4.75145
3 4.65625
4 4.66824
5 4.66874

We see very good convergence to the Feigenbaum ratio 4.6692... .
Qu. 7

(a)

\[ f(x) := \lambda \sin(\pi x) \]

\[
\text{lya}[\lambda, \text{xinit}, n, \text{ndrop}] := (\lambda = 1; \\
\text{xlist} = \text{Drop}[\text{NestList}[f, \text{xinit}, n], \text{ndrop} + 1]; \text{Apply}[\text{Plus}, \text{Log}[\text{Abs}[f'[\text{xlist}]]]/\text{Length}[\text{xlist}]) \\
\]

\[
\text{lya}[0.9, 0.4, 10000, 1000] \\
0.35973
\]

The Lyapunov is positive so we are in the chaotic regime.

(b)

Since we are in the chaotic regime we need to use high-precision numbers to get an accurate result for \(x_{5000}\) starting from \(x_0 = 4/10\). Let’s write \(\lambda\) exactly

\[ \lambda = 9/10; \]

and make an initial guess that 1000 digits will be enough by specifying \(x_0\) to 1000 digits

\[ x_0 = \text{N}[4/10, 1000]; \]

The we do the iterations and compute the precision.

\[ \text{ans} = \text{Nest}[f, x_0, 5000]; \]

\[ \text{Precision}[\text{ans}] \]

251.78

This is plenty of precision so we print the result.

\[ \text{N}[\text{ans}] \]

0.795585

Note that if we use default precision we get a quite wrong answer

\[ \text{Nest}[f, 0.4, 5000] \]

0.899696