

PHYSICS 115/242

Homework 8

To be handed in at the last class on Friday, June 6.

Final Exam: The final will be a take home exam. The exam can be picked up in class on Friday June 6, or downloaded from my web site. It must be returned to my mailbox in ISB 232, by 5 pm. Tuesday June 10. The final must be your own work. *No collaboration or discussion with others is allowed for the final.*

1. *The Mandelbrot Set, and its relationship to the logistic map*

- (a) There is a close connection between the map used to determine the Mandelbrot set

$$z' = z^2 + c,$$

and the logistic map

$$x' = 4\lambda x(1 - x).$$

Take c and z to be real, write $z_n = ax_n + b$, and show that these maps are equivalent for a certain choice of a and b provided there is a relationship, which you should determine, between λ and c .

- (b) As λ varies from $1/4$ to 1 , over what range does c vary?
(c) In the Mandelbrot one conventionally starts the map at $z = 0$. What is the corresponding value of x in the logistic map?
(d) Produce a graph of the Mandelbrot set.
(e) What feature of the Mandelbrot set is traversed as c varies from 0.25 to -0.75 ? What is the behavior of the logistic map in the corresponding range? What do you expect is the limiting behavior of the z_n in the Mandelbrot set for c inside the big cardioid?
(f) Going along the real axis to the left of the big cardioid, there is a succession of “blobs” getting smaller and smaller until they disappear. What features of the logistic map correspond to these blobs?
(g) There is another blob in the Mandelbrot set on the real axis at c around -1.76 . What features of the logistic map do this correspond to?

2. *The Lorenz Equations*

Consider the Lorenz equations, which were originally written down by Lorenz (not the Lorentz as in “Lorentz Force” or “Lorentz Contraction”) in 1963 as a model for convection. These equations became famous because Lorenz realized that for certain ranges of parameters the motion is chaotic. The equations are:

$$\begin{aligned}x'(t) &= -\sigma x + \sigma y \\y'(t) &= -xz + rx - y \\z'(t) &= xy - bz,\end{aligned}$$

where $x(t)$, $y(t)$ and $z(t)$ are the three variables to be solved for, and r , σ and b are parameters. In this question take $r = 26.5$, $\sigma = 10$, and $b = 8/3$, and start the system off with $x(0) = 1$, $y(0) = 0$, $z(0) = 0$.

- (a) Display the trajectory in the three-dimensional space, $x-y-z$, using the Mathematica command `ParametricPlot3D`.
- (b) Make a graph of each of the three variables against t . Do the solutions appear chaotic?

3. Damped Driven Oscillator

Consider the damped driven pendulum, which is discussed in Tam, p. 356-9. The equation of motion is

$$\frac{d\omega}{dt} = -\sin \theta - a\omega + f \cos \Omega t,$$

where θ is the angle of the pendulum, $\omega \equiv d\theta/dt$ is the *angular* frequency, a is a damping coefficient, f is the amplitude of the driving force, and Ω is its angular frequency.

Note: Since θ is an angle, when you plot it you should always bring it back in to the range from $-\pi$ to π .

- (a) Find parameters with periodic motion and demonstrate that the motion is periodic by displaying the Poincaré section.
- (b) Find parameters with chaotic motion and demonstrate that the motion is chaotic by displaying the Poincaré section.

Hint: To get ideas for possible parameters you could look at Tam, “A physicist’s guide to Mathematica”, p. 359–60.

4. The Koch Curve

Consider the Koch curve formed by taking a line and forming a “kink” in it as shown.



Each of the resulting four line segments are of equal length and the angles are 60 degrees. Then repeat the process for each of the resulting line segments, and then for each of the (even smaller) segments, and so on.

- (a) What is the fractal dimension of the curve?
- (b) Assume that the initial length of the line is 1. What is the length of the curve after 1 iteration? after 2 iterations? after n iterations? after an infinite number of iterations?
- (c) Produce a graph of the curve.
- (d) Start with an equilateral triangle and apply the transformation to each side to obtain the “Koch snowflake”. Produce a graph of this.

Note: Like real snowflakes, it should have 6-fold symmetry.

5. A Quantum Well

Consider a particle in a well with the following potential:

$$V(x) = -\frac{V_0}{2} \left[1 + \cos\left(\frac{2\pi x}{L}\right) \right] \quad (|x| < L/2),$$

and $V(x) = 0$ otherwise. We will take $V_0 = 25$ and $L = 1$, as well as $\hbar = m = 1$.

- (a) Plot the potential well.
- (b) Find the energy of the ground state. Also plot the corresponding (normalized) wavefunction.
- (c) Find the energy of lowest odd-parity state. Also plot the corresponding (normalized) wavefunction.
- (d) *Physics 242 students only.*
Are there any more bound states, other than the two you found in parts (b) and (c)?
Estimate the value of V_0 needed for a second bound state with even parity to appear.

6. *Another Quantum Well*

Consider a particle in a well with the following potential

$$V(x) = -\operatorname{sech}^2(x),$$

in units where $\hbar = m = 1$. In order to use the technique discussed in the class, assume that $V(x) = 0$ for $|x| > 5$. (The resulting discontinuity is tiny and negligible.)

- (a) Plot the potential well.
 - (b) Find the energy of the ground state. Also plot the corresponding (normalized) wavefunction.
 - (c) You are *given* that the exact ground state wavefunction is $\psi(x) = (1/\sqrt{2}) \operatorname{sech}(x)$. What is the exact energy eigenvalue? Compare your wavefunction and energy eigenvalue with the exact results.
7. *The shooting method. Physics 242 students only.*

Consider the anharmonic oscillator

$$\mathcal{H} = \frac{p^2}{2} + \frac{x^4}{4}.$$

Starting from Schrödinger's equation, use the "shooting method" discussed in class to determine the lowest three energy levels. (Take $\hbar = 1$.)

8. *Matrix methods. Physics 242 students only.* Consider the same anharmonic oscillator

$$\mathcal{H} = \frac{p^2}{2} + \frac{x^4}{4},$$

as in Qu. 7, (note that there is *no* x^2 term in the potential energy). Working in units where $\hbar = 1$, determine the first three energy levels using the *matrix* methods discussed in class.

Compare your results with those from the shooting method in Qu. 7.