

PHYSICS 112

Final Exam, 2012, Tuesday March 20, 12:00–3:00 pm

**Closed book.** You may bring one sheet of notes you have prepared yourself.

**You must explain your reasoning.**

Calculators, or other electronic devices, are **not** allowed. Cell phones must be switched off.

1. [20 points]

Consider a simple model with two Ising spins, each of which takes values  $\pm 1$ . The energy is given by

$$E = -JS_1S_2.$$

- (a) Enumerate the possible states and their energy.
- (b) Determine the partition function and free energy.
- (c) Show that

$$\langle S_1S_2 \rangle = \tanh \beta J,$$

where  $\beta = 1/k_B T$ .

- (d) What is  $\langle S_1 \rangle$ ?

2. [18 points]

Consider an impurity atom in a semiconductor. The atom may lose an electron to the conduction band of the semiconductor crystal. Let the ionization energy of the impurity atom be  $I$  and we initially assume that only one electron can be bound to the impurity. Hence the atom has three states:

State number	Description	$N$	$\epsilon$
1	Electron detached	0	0
2	Electron attached, spin $\uparrow$	1	$-I$
3	Electron attached, spin $\downarrow$	1	$-I$

We have taken the zero of energy to be that of the impurity with no electrons.

- (a) Calculate the probability that an impurity atom is ionized (i.e. has no electrons).  
*Note:* Your answer will involve the chemical potential of the electrons.
- (b) Suppose now that the impurity atom can bind two electrons (one of each spin) into a state with energy  $-2I+U$  ( $< 0$ ), where  $U$  arises from the Coulomb repulsion between the electrons. What is the probability that an impurity atom binds two electrons?
- (c) Simplify the expression in Qu. 2b for the case of  $U = 0$  and interpret your result.

3. [12 points]

Consider an elastic band under tension  $f$ . If the length of the band is  $l$  the thermodynamic identity can be expressed as

$$dF = -S dT + f dl,$$

where  $F$  is the free energy.

- (a) Derive the Maxwell relation

$$\left(\frac{\partial f}{\partial T}\right)_l = -\left(\frac{\partial S}{\partial l}\right)_T.$$

(b) Hence use the third law of thermodynamics to show that

$$\left(\frac{\partial f}{\partial T}\right)_l \rightarrow 0$$

as  $T \rightarrow 0$  (i.e. if the length is fixed the tension becomes independent of  $T$  in the low- $T$  limit).

4. [25 points]

As we have discussed during the course, the density of states of free particles in *two dimensions* is a constant, i.e. independent of energy. More precisely, for spinless bosons you are *given* that

$$\rho(\epsilon) = A \frac{m}{2\pi\hbar^2},$$

where  $A$  is the area of the system (no need to derive this). The number of bosons is  $N$ .

(a) Show that

$$n \equiv \frac{N}{A} = n_Q \int_{-\beta\mu}^{\infty} \frac{dx}{e^x - 1},$$

where

$$n_Q = \frac{mk_B T}{2\pi\hbar^2}$$

is the *two-dimensional* quantum concentration.

(b) Do the integral by writing

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}}$$

and hence determine an *explicit* expression for  $\mu$  in terms of  $n$ ,  $n_Q$  and  $T$ .

*Note:* It is possible to get an explicit expression for the temperature dependence of the chemical potential in two dimensions (though not in other dimensions) because the density of states is particularly simple (a constant).

(c) Use your result to argue that there is *no* Bose-Einstein condensation in Bose gas in two dimensions, but the chemical potential becomes exponentially small in  $T$  as  $T \rightarrow 0$ .

(d) Show that for large  $T$  one recovers the result for the classical ideal gas in two dimensions:

$$\mu = -k_B T \ln\left(\frac{n_Q}{n}\right).$$

5. [25 points]

Consider an Ising model in which the spin  $S$  takes five possible values  $\pm 2, \pm 1$  and  $0$ .

(a) For a single spin in a magnetic field with energy given by

$$E = -BS,$$

show that  $\langle S \rangle$  is given by

$$\langle S \rangle = f\left(\frac{B}{k_B T}\right),$$

where  $f(x)$  is a function which you should calculate.

(b) Show that for small  $x$ ,

$$f(x) = 2x + \dots$$

- (c) Now assume that  $B = 0$  and there are spins on the sites of a crystal lattice in which each site has  $z$  neighbors. The energy (for  $B = 0$ ) is given by

$$E = -J \sum_{\langle i,j \rangle} S_i S_j,$$

where the sum is over nearest neighbor pairs. We want to calculate average of the spin at site  $i$ . In the mean field approximation, to do this we replace the spin  $S_j$  at a neighboring site  $j$  by its *average* value  $\langle S_j \rangle \equiv m$ . Hence show that, in mean field theory, the average value of a spin,  $m$ , is given by the self-consistent equation

$$m = f\left(\frac{zJm}{k_B T}\right), \tag{1}$$

where  $f(x)$  is the same function as in part (5a).

- (d) Assuming that the transition is second order, so the transition temperature  $T_c$  is where  $m \rightarrow 0$  and hence is where the coefficients of  $m$  on both sides of Eq. (1) are equal, determine the value of  $T_c$  in the mean field approximation.