(a) There are 4 states per impurity level, for energy $E_0$ # of electrons

\[
\begin{aligned}
E_d &= 0 \\
E_{d/2} &= 2 \\
2E_d + \Delta &= 2 \\
2E_{d/2} + \Delta &= 2
\end{aligned}
\]

Hence

\[
\frac{n_d}{N_d} = \frac{0 + 2e^{-\beta(E_d-M)} - 2e^{-(2E_d-E_d+\Delta)}}{1 + 2e^{-\beta(E_d-A)} + e^{-\beta(2E_d-E_d+\Delta)}}
\]

\[
= \frac{1 + e^{-\beta(E_d-M)}}{1 + \frac{1}{2} e^{\beta(E_d-M)} + \frac{1}{2} e^{-\beta(E_d-M)}}
\]

\[
= \frac{2}{e^{\beta(E_d-M)} + 1}
\]

i.e. twice the Fermi function as expected.

For $\Delta \to \infty$

\[
\frac{n_d}{N_d} = \frac{1}{1 + \frac{1}{2} e^{\beta(E_d-M)}}
\]

\[
\Delta \to \infty \quad \text{Eq. (28.32)}
\]

(b) For $\Delta \to 0$

\[
\frac{n_d}{N_d} = \frac{1 + e^{-\beta(E_d-M)}}{1 + \frac{1}{2} e^{\beta(E_d-M)} + \frac{1}{2} e^{-\beta(E_d-M)}}
\]

\[
= 2 \frac{1 + e^{-\beta(E_d-M)}}{[1 + e^{\beta(E_d-M)}][1 + e^{-\beta(E_d-M)}]} = \frac{2}{e^{\beta(E_d-M)} + 1}
\]

(c) Assume that only 1 electron is present in any of the orbit. Then, the states are:

- $E_c$ (1 electron)
- or 0 electrons

Hence

\[
\frac{n_d}{N_d} = \frac{2 \sum_i e^{-\beta(E_i-M)}}{1 + 2 \sum_i e^{-\beta(E_i-M)}}
\]

Factors of 2 because there are 2 states with energy $E_i$ because of spin degeneracy.
1. (b) Very little will change. Except at low $T$ the 
impurities are fully ionized. This remains the same.
At low $T$ the fraction of bound impurities is exponentially small. 
Thus remains qualitatively the same.

$$E_d = \left( \frac{m_e}{m} \right)^{1/2} x 13.6 \text{ eV} = 6.3 \times 10^{-4} \text{ eV}$$

$$\gamma_0 = \frac{m_e}{m} \times 0.529 \text{ A} = 635 \text{ A}$$

(c) Need mean spacing to be about $\gamma_0$

i.e. $\frac{1}{N_d} \approx \frac{4 \pi}{5} \gamma_0^3 \Rightarrow N_d \approx 10^{15} / \text{cm}^3$

3. (a) In $Si$ there are 6 pockets along the $<100>$ directions.
The field is in a direction

$$\left( \frac{\sin \theta}{12}, \frac{\sin \phi}{12}, \cos \phi \right)$$

with $\phi = 30^\circ$

The effective mass is the geometric mean of the 
band masses in directions $1$ to the field, see Q. 1 of 
Hw. 5.

From Kittel Ch. I (Eq. 134) in my edition) this leads

$$\left( \frac{1}{m^*} \right)^2 = \frac{\cos^2 \phi}{m_1^2} + \frac{\sin^2 \phi}{m_2^2}$$

$$m_1 m_2$$

where $\phi$ is the angle between the field and the 
direction of the major axis of the ellipsoid of constant 
energy, $m_1$ is the longitudinal effective mass, and $m_2$ is 
the transverse effective mass.

For the pockets along $(0, 0, 1)$ and $(1, 0, 0)$ $\phi = \gamma = 630^\circ$

For the pockets along $(1, 0, 0)$ and $(1, \pm 1, 0)$

$$\cos \phi = \frac{1}{12} \sin \phi = \frac{1}{12} \sin 1 \Rightarrow 0 \neq \phi = 52^\circ 69.3^\circ$$

Because there are 2 values of $\phi$ there are 2 values for 
the resonance frequency.
(b) With \( m_c = m \),

\[
\nu = \frac{\omega}{2\pi} = \frac{eH}{2\pi m_c} = \frac{9.1 \times 10^{-28} \times 4.8 \times 10^{-10}}{6.2 \times 9.1 \times 10^{-37} \times 3 \times 10^{10}} \frac{Hm}{m_c} = 2.8 \times 10^{10} \frac{m}{m_c}
\]

if \( \nu = 2.4 \times 10^{10} \)

\[
H = \frac{m_c \times 2.4 \times 10^{10}}{2.8 \times 10^{10}} = 0.86 \times 10^{-4} \frac{m}{m} \text{ Gauss}
\]

With \( m_L \sim m, m_c \sim \frac{0.2}{m}, \theta \approx 30^\circ \)

\[
\frac{m^2}{m_c^2} = \frac{3}{4} \frac{1}{0.2^2} + \frac{1}{4} \frac{1}{0.2} = 20.
\]

\( 20 m_c = 0.22 \), \( \Rightarrow H = 8.6 \times 10^3 \text{ Gauss} \)

This is the lower electron peak in Fig. 28.9.

With \( m_c \sim m, m_t \sim 0.2m, \theta \approx 69.3^\circ \)

\[
\frac{(m_c)}{m} = \frac{0.125 + \frac{2 \times 0.2}{0.2}}{8 \times 0.2} = 7.5, \Rightarrow m_c \sim 0.365
\]

\( \Rightarrow H = 8.6 \times 10^3 \times 0.365 = 3.14 \times 10^3 \text{ Gauss} \)

This is the upper electron peak in Fig. 28.9.
4. (a) \[ \bigcup_{V} \epsilon_{d} \bigcup_{A} \epsilon_{a} \bigcup_{V} \epsilon_{v} \] 

At \( T = 0 \), \( n_{c} = 0 \) \( p_{v} = 0 \)

Na electrons drop from the donor levels to fill the acceptor levels

\[ \Rightarrow n_{d} = N_{d} - p_{a}, \quad p_{a} = 0 \]

Note that \( n_{d} - N_{d} = -p_{a} \) or \( n_{c} + n_{d} - N_{d} = p_{v} + p_{a} - p_{a} \)

(b) Eq. 1 above still holds at \( T = 0 \) because each excitation of an electron creates a hole (and so increases \( p_{v} \) or \( p_{a} \) by 1) and on electron (and so increases \( p_{c} \) or \( n_{a} \) by 1).

(c) At \( T = 0 \) \( n_{c} = p_{v} = 0 \)

\[ n_{d} = 0, \quad p_{a} = p_{a} - N_{d}. \]

(d) Same logic as for part (b).

5. (a) From Eq. (4) with \( P_{a} = p_{a} = 0 \) (since there are no acceptors) and \( P_{v} = 0 \) (since we are told that no electrons are excited from the valence bond), we have:

\[ N_{c}(T) = N_{d} - N_{d}(T) \]

(b) From the discussion in Eq. (1) we have:

\[ n_{c}(T) = \frac{N_{d}}{1 + \frac{1}{2} e^{-\frac{\left( \epsilon_{c} - \mu \right)}{k_{B}T}}} \]

and, using standard book work:

\[ n_{c}(T) = e^{-\beta \left( \epsilon_{c} - \mu \right)} N_{c}(T) \] where \( N_{c}(T) = \frac{1}{4} \left( \frac{2m_{e}k_{B}T}{\pi \hbar^{2}} \right)^{\frac{3}{2}} \)}
So \(N_c(T) e^{-\beta (E_c - \lambda)} = N_d \left[ 1 - \frac{1}{\frac{1}{2} e^{\beta(E_c - \lambda)} + 1} \right]\)

This is a quadratic equation for \(\lambda = e^{\beta E_d}\).

\[
\frac{N_c e^{-\beta E_c}}{N_d} \lambda = 1 - \frac{1}{\frac{1}{2} e^{\beta E_d} \lambda + 1}
\]

\[
\Rightarrow \frac{N_c}{N_d} e^{-\beta E_c} \lambda = \frac{1}{2} e^{\beta E_d} \lambda + \frac{1}{2} e^{\beta E_d}
\]

\[
\lambda \left( \lambda + \frac{1}{2} e^{\beta E_d} \right) = \frac{1}{2} e^{2\beta E_d}
\]

\[
\lambda = -\frac{1}{2} e^{\beta E_d} \pm \sqrt{\frac{1}{4} e^{2\beta E_d} + 2 \frac{1}{2} e^{\beta E_d}}
\]

\[
\lambda = 2 e^{\beta E_d} \left[ \frac{1 + \sqrt{1 + 8x}}{4} \right]
\]

\[
\eta(T) = E_d + k_B T \ln \left[ \frac{1 + \sqrt{1 + 8x(T)}}{4} \right]
\]

(Note: \(E_c\) energy scale, \(T_d\) cut-off temperature, \(N_c(7)\) set by the impurity density, and \(E_c - E_d\) set by the impurity energy.)

Now \(E_c \geq E_d\) and \(\lambda \sim T^{-3/2}\)

\(x(7)\) is a monotonically decreasing function of \(T\).

\(x(7) = 1\), then defines \(T_d\). Note that \(T_d > T_d\).

When \(T > T_d\), \(\exp(\beta E_d) \ll 1\)

\[\ln \left[ \frac{1 + \sqrt{1 + 8x}}{4} \right] = \ln \left[ \frac{1 + 1 - 4x}{4} \right] = \ln x \]

\[\ln x\] which is large and negative.

Hence \(\eta(T) - E_d\) is large and negative, proportionally.

\[
\eta(T) = \frac{\eta(T)}{k_B T}
\]

\[
\eta(T) = \left( E_c - E_d \right) - \frac{3}{2} k_B T \ln \left( T/T_d \right)
\]

Thus \(N_d(T) \ll 1\) and \(N_c(T) = N_d\) from part (a).
(e) For $T < T_0$, i.e. $\delta T (T) \gg 1$, \[ \mu (T) = \frac{\varepsilon_0}{2} \left( \varepsilon_c - \varepsilon_0 \right) + \frac{k_b T}{2} \ln \left( \frac{N_d}{2N_c(T)} \right) \]
\[ = \left( \frac{\varepsilon_c + \varepsilon_0}{2} \right) + \frac{k_b T}{2} \ln \left( \frac{N_d}{2N_c(T)} \right) = \left( \frac{\varepsilon_c + \varepsilon_0}{2} \right) + k_b T \left[ \frac{3}{2} \ln (\frac{T}{T_0}) - \frac{1}{2} \right] \]

(i.e. for $T \rightarrow 0$, $\mu$ is halfway in between $\varepsilon_c$ and $\varepsilon_0$)

From Eq. (6) and part (6)
\[ n_c(T) = N_c(T) e^{-\beta (\varepsilon_c - \mu)} = \sqrt{\frac{N_c(T) N_d}{2}} e^{-\beta (\varepsilon_c - \varepsilon_0) / 2} = \frac{N_d}{\sqrt{12}} \left( \frac{T}{T_0} \right)^{3/2} e^{-\beta (\varepsilon_c - \varepsilon_0) / 2} \]

(6) \( T = 300 \text{K} \)
\[ \varepsilon_c - \varepsilon_0 = 2 \text{meV} \approx 2.3 \text{eV} \]
\[ 2.5 \left( \frac{m_e}{m_i} \right)^{1/2} \left( \frac{T}{300} \right)^{3/2} \approx 1.6 \times 10^{19} \text{cm}^{-3} \]

From AM
\[ \approx 2.5 \times 10^{16} \text{cm}^{-3} \]
\[ \approx 2.5 \times 10^{16} \leq 1 \ll 1, \text{ i.e. the high-} T \text{ regime.} \]

\[ \approx 7 \text{N}_c(T) \approx N_d = 10^{15} \text{cm}^{-3} \]

(11) \( T = 41 \text{K} \)
\[ N_c(T) = 2.5 \left( \frac{4}{300} \right)^{3/2} \times 10^{16} \approx 3.85 \times 10^{13} \]
\[ \Rightarrow \delta N = \frac{10^{15}}{3.85 \times 10^{13}} e^{-\beta (\varepsilon_c - \varepsilon_0) / 2}, \text{ i.e. the low-} T \text{ regime} \]
\[ N_c(T) = \sqrt{\frac{N_c(T) N_d}{2}} e^{-\beta (\varepsilon_c - \varepsilon_0) / 2} \approx 7.6 \times 10^{12} \text{cm}^{-3} \]

i.e. much less than in part (i)

Note: For comparison, $T_d$ is given by $N_c(T_d) = N_d$
\[ \text{i.e.} \ 2.5 \left( \frac{m_e}{m_i} \right)^{1/2} \left( \frac{T_d}{300} \right)^{3/2} \times 10^{19} = 10^{15}, \Rightarrow \left( \frac{T_d}{300} \right)^{3/2} = \frac{1}{25} \]
\[ \Rightarrow T_d = 35 \text{K}. \text{ Solving Eq. (11) for Td gives } \Rightarrow T_d = 48 \text{ K.} \]