Intensity of Bragg scattering is proportional to \( e^{-2W} \)

where \( W = \frac{1}{N} \sum \frac{k}{2MW_{ik}} (\alpha_i - \xi_{ik})^2 \left[ N \left( \frac{k}{2MW_{ik}} \right) + \frac{1}{2} \right] \)

Averaging over deviations, which gives an unimportant numerical factor of order unity, and replacing the sum by an integral gives

\[ W \propto \int_0^\infty dk \, k^{d-1} \left[ \frac{1}{k} \int_0^{\infty} \frac{dt}{e^{st} - 1} \right] \]

Now \( W \propto k^d \) and we are interested in the divergent behavior of the integrand as \( k \to 0 \), so

\[ W \propto \int_0^\infty dk \, k^{d-1} \left[ \frac{1}{k} \right] \]

For \( T > 0 \) the \( T/k \) term dominates over \( 1/k \)

\[ W \propto T \int_0^\infty \frac{dk}{k^2} \]

- \( d = 2 \) \[ W \propto T \int_0^\infty \frac{dk}{k} \]
- \( d = 1 \) \[ W \propto T \int_0^\infty \frac{dk}{k^2} \]
- \( d = 3 \) \[ W \text{ is finite.} \]

(b) \( T = 0 \)

\[ W \propto \int_0^\infty \frac{dk}{k^2} \]

- \( d = 2 \) \[ W \propto \int_0^\infty \frac{dk}{k^{d-1}} \text{ const. which is finite.} \]
- \( d = 1 \) \[ W \propto \int_0^\infty \frac{dk}{k} \text{ which diverges logarithmically.} \]
(c) If \( W = \infty \), \( e^{-2W} = 0 \) so there is no translational long range order when \( W \) diverges (though one has "quasi long range order" in 2-d (at finite \( -T \)), see the handout.

3. Dispersion relation
\[
\frac{d^2 W}{dk^2} < 0
\]
3 phonon processes
(a) all phonons belong to the same branch.

Graphical construction:

(b) Since \( W(k) \) is a process
\[
L + L \leftrightarrow T
\]
also does not work because the mismatch in frequencies (if \( k \) conservation is satisfied) is even greater.

Similarly for \( L + T \leftrightarrow T \).

In order to make up the mismatch in frequencies, the single phonon must be on a branch higher than at least one of the members of the pair, e.g.
\[ T + I \iff L \]  

Similarly \( T + L \iff L \) works.

3. (a) Density of points in \( k \)-space/unit volume is \( \frac{1}{(2\pi)^2} \)

Hence \( n = \frac{2}{\pi} \frac{\pi k_F^2}{(2\pi)^2} = \frac{k_F^2}{2\pi} \)

\[ \frac{1}{(2\pi)^2} \text{ is the area per particle.} \]

(b) \( n = \frac{1}{\pi k_F^2} \)

\( \therefore \frac{1}{\pi k_F^2} = \frac{k_F^2}{2\pi} \text{ or } k_F = \frac{\sqrt{2}}{\sqrt{\pi}} \)

(c) \( g(\varepsilon) \, d\varepsilon = \frac{2}{\pi} \frac{2\pi k \, dk}{\text{area of shell}} \frac{1}{(2\pi)^2} \)

\[ g(\varepsilon) = \frac{k}{\pi} \frac{1}{\varepsilon^2} \text{ for } \varepsilon > 0 \]

8. (d) Sommerfeld expansion

\[ n = \int_0^\infty f(\varepsilon) g(\varepsilon) \, d\varepsilon = \int_0^\infty \frac{\varepsilon^2}{n-1} \frac{\partial^n}{\partial \varepsilon^n} g(\varepsilon) \, d\varepsilon \]

But all derivatives vanish so

\[ n = \int_0^\infty g(\varepsilon) \, d\varepsilon \]

At \( T = 0 \),

\[ n = \int_0^\infty \delta(\varepsilon) \, d\varepsilon \]

According to the Sommerfeld expansion.
(e) Does this mean that \( \mu(T) \) is really identical to \( E_F(\equiv \mu(T=0)) \)?

No. It just means that all terms that are powers of \( T \) vanish. As we shall see in the next section there are corrections which vanish exponentially as \( T \to 0 \).

Now \( n = \frac{m}{\pi^2 k^2} \int_0^{E_F} dE = \frac{m E_F}{\pi^2 k^2} 1 \quad (T=0) \) and \( n = \frac{m}{\pi^2 k^2} \int_0^{\infty} \frac{e^{-1}}{e^{E/kT} + 1} dE = \frac{mkB_7}{\pi^2 k^2} \int_0^{\infty} \frac{dx}{e^{x/kT} + 1} \quad (T>0) \)

Now \( I = \int \frac{dx}{e^{x/kT} + 1} = \int \left[ 1 - \frac{e^{-x/kT}}{e^{x/kT} + 1} \right] dx = x - \ln(1 + e^{x/kT}) = 0 \)

Now \( n = \frac{m k_B T}{\pi^2 k^2} \left[ \ln \left( \frac{e^{x/kT}}{k_B T} \right) + \ln \left( 1 + e^{x/kT} \right) \right] \quad (2) \)


\[
E_F = \mu + k_B T \ln \left( 1 + e^{-\mu/k_B T} \right)
\]

Equate (1) and (2) \( \ln[1 + e^{\mu/k_B T}] \), so from (1) and (2) \( \frac{e^{B_7 T} - e^{B_7 T - 1}}{e^{B_7 T} - 1} \)

\( \mu = k_B T \ln \left( \frac{e^{B_7 T} - 1}{B_7} \right) \)

(f) As \( T \to 0 \), \( \mu \) is very close to \( E_F \). Also, \( e^{-\mu/k_B T} \) is very small and so, to leading order in small quantities, we can replace it by \( e^{-\mu/k_B T} \).

Hence \( \mu(T) \approx E_F - k_B T \ln \left( 1 + e^{\mu/k_B T} \right) \approx E_F - \frac{E_F}{e^{\mu/k_B T}} \)

\( \approx E_F - k_B T \quad - \quad - \approx \)

i.e. \( \mu - E_F \) is exponentially small as \( T \to 0 \)

(9) (Note that this is Eq.)

at high \( T \) from (3) \( \mu = k_B T \ln (B_7 F) = k_B T \ln \left( \frac{n \pi^2 \lambda^2}{mk_B T} \right) \quad ( \geq 0) \)

\[
= k_B T \ln (n A_0) \quad A_0 = \frac{\lambda^2}{mk_B T} \quad \text{the square of the Fermi wavelength}
\]
Use the classical expression

\[ n = \int_0^\infty e^{-\left(\frac{E-x}{k_B T}\right)} g(x) \, dx. \]

Now \( g(x) = \frac{\sqrt{2}}{\pi} \left(\frac{m}{k^2}\right)^{3/2} e^{-x/2} \geq 0 \)

\[ n = \frac{\sqrt{2}}{\pi} \left(\frac{m}{k^2}\right)^{3/2} e^{-x} \left(\frac{1}{k_B T}\right)^{3/2} \int_0^\infty e^{-x/2} e^{-x} \, dx \]

\[ \therefore n = \frac{1}{\sqrt{2}} \left(\frac{m}{k^2 k_B T}\right)^{3/2} e^{-x} \]

But this is only valid if \( e^{x/k_B T} \ll 1 \)

i.e. \( n \ll \frac{1}{\sqrt{2}} \left(\frac{m}{k^2 k_B T}\right)^{3/2} \)

But \( n = \frac{4 \pi r_s^3}{3} \), \( r_s \) is the typical inter-particle spacing

\[ r_s \gg \left(\frac{k^2}{2mk_B T}\right)^{1/2}. \]

(1) (dropping factors of order unity)

\[ \left(\frac{\hbar}{2mk_B T}\right)^{1/2} \] is the thermal de-Broglie wavelength

i.e. the de-Broglie wavelength of a particle whose energy \( \sim k_B T \) (times \( \frac{1}{\hbar} \))

\[ \Rightarrow \frac{k^2}{2m} \left(\frac{2\pi \hbar}{\lambda_{\text{de-B}}}\right)^2 = k_B T \]

(c) just plug in the numbers See next page
5. \( V_F = \frac{k}{m} k_F \), where \( k_0 \) is.

Roughly, \( k_F \sim \frac{1}{a_0} \), the Bohr radius (i.e. of order the interatomic separation).

\[ a_0 = \frac{\hbar^2}{me^2} \]

Hence \( V_F \leq \frac{k}{m} \frac{me^2}{\hbar} = \frac{e^2}{\hbar c} \simeq \frac{e^2}{\hbar c} \]

where \( \lambda = \frac{e^2}{\hbar c} \simeq \frac{1}{157} \) is the fine structure constant.

(6) The cores are roughly 4 orders of magnitude heavier than the electrons and, in general, \( \omega \sim m^{-\frac{1}{2}} \) (see e.g. the simple harmonic oscillator).

4 (c) \( \hbar = 1.05 \times 10^{-34} \), \( m = 9.1 \times 10^{-31} \), \( \hbar = 1.29 \times 10^{-25} \)

\[ \frac{a_0}{a_0} = 0.53 \times 10^{-10} \]

\[ \text{Hence} \quad \frac{k}{a_0} \approx \left( \frac{\hbar}{2m k_B T} a_0 \right)^{1/2} = \left[ \frac{(1.05 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31} \times 1.37 \times 10^{-23} \times (0.53 \times 10^{-10})^2} \right]^{1/2} \]

\[ = \left[ \frac{1.6 \times 10^5}{1} \right]^{1/2} \approx \left[ \frac{1.05}{1} \right]^{1/2} \]