1. The probabilities follow the Poisson distribution

\[ P(n) = \frac{\mu^n}{n!} e^{-\mu}, \]

with \( \mu = 5 \). Plugging in the numbers gives

(a) \[ P(0) = 0.00674. \]

(b) \[ P(1) = 0.0337. \]

(c) \[ P(5) = 0.1755. \]

(d) \[ P(20) = 2.64 \times 10^{-7}. \]

Note: The probability is peaked in the vicinity of the mean.

2. (a) Let \( x_i = 1 \) if I win at attempt \( i \), and \( x_i = 0 \) if I lose. Then, assuming that the odds are 50-50, we have

\[ \langle x \rangle = \frac{1}{2}, \quad \sigma_x = \frac{1}{2}. \]

Writing

\[ \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \]

we have, following the results derived in class for the mean and standard deviation of a sum of random variables,

\[ \langle \bar{x} \rangle = \frac{1}{2}, \quad \sigma_{\bar{x}} = \frac{1}{2\sqrt{N}} = \frac{1}{20} = 0.05. \]

Since I win a fraction 0.3 of the attempts, compared with supposed average of 0.5, the difference is

\[ \frac{0.5 - 0.3}{\sigma_{\bar{x}}} = 4 \]

standard deviations.

(b) The probability of the result deviating by more than \( \pm k \) standard deviations for a Gaussian distribution is given by

\[ 2 \frac{1}{\sqrt{2\pi}} \int_k^{\infty} e^{-x^2/2} dx = 2 \frac{1}{\sqrt{\pi}} \int_{k/\sqrt{2}}^{\infty} e^{-t^2} dt = \text{erfc}(k/\sqrt{2}). \]

Here we have \( k = 4 \) and so the probability is \[ \text{erfc}(4/\sqrt{2}) = 0.00006334 \]

(c) No!
3. The Fourier transform of a Lorentzian is given by

\[ g(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + x^2} \, dx = e^{-|k|}, \]

where we performed the integral by contour integration. This was done in the Physics 116B class and in Qu. 1 of HW 8 of this class. See also the handout for this class on “Some comments on contour integrals.” Hence the Fourier transform of the distribution of the sum of \( N \) variables is given by

\[ G_N(k) = g(k)^N = e^{-N|k|}. \]

Hence the distribution of the sum is given by

\[ \tilde{P}_N(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-N|k|} e^{-ikX} \, dk = \frac{1}{2\pi} \int_{-\infty}^{0} e^{Nk} e^{-ikX} \, dk + \frac{1}{2\pi} \int_{0}^{\infty} e^{-Nk} e^{-ikX} \, dk \]

\[ = \frac{1}{2\pi} \left[ \frac{1}{N - iX} + \frac{1}{N + iX} \right] = \frac{1}{N\pi} \frac{1}{1 + (X/N)^2}, \]

Writing this in terms of the sample mean \( \bar{x} = X/N \), and noting that \( \tilde{P}_N(X) \, dX = P_N(\bar{x}) \, d\bar{x} \) (in order that both distributions are normalized to unity) where \( dX/d\bar{x} = N \) we have

\[ P_N(\bar{x}) = \frac{1}{\pi} \frac{1}{1 + (\bar{x}/N)^2}, \]

which is the same as the original Lorentzian distribution.

Note: For a Gaussian distribution (or any other distribution with a finite variance, \( \sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 \), for which the “central limit theorem” holds) the distribution of the sample mean is narrower than the distribution of a single data point (by a factor of order \( 1/\sqrt{N} \)). Hence, in these cases, for large \( N \), the sample mean is very close to the exact mean of the distribution. Furthermore, in these cases, the distribution of the sample mean is Gaussian. However, here, the variance is not finite, the central limit theorem does not hold, and the distribution of the sample mean is not Gaussian (it is Lorentzian) and is just as broad as the distribution of \( x \). The reason is that, since the distribution only falls off slowly at large \( x \), there is a significant probability of getting in the sample some values which are much larger, in magnitude, than the typical value, and these dominate the sample mean.

4. We consider the exponential distribution

\[ P(x) = \begin{cases} \frac{1}{x_0} \exp(-x/x_0), & (x \geq 0), \\ 0, & (x < 0). \end{cases} \]

(a) The mean is given by

\[ \langle x \rangle = \frac{1}{x_0} \int_{0}^{\infty} x \exp(-x/x_0) \, dx 
= x_0 \int_{0}^{\infty} t \exp(-t) \, dt 
= x_0 \Gamma(2) = \frac{x_0}{2}, \]
where $\Gamma(x)$ is the Euler Gamma function. Similarly the average of $x^2$ is given by

$$\langle x^2 \rangle = \frac{1}{x_0} \int_0^\infty x^2 \exp(-x/x_0) \, dx$$

$$= x_0^2 \int_0^\infty t^2 \exp(-t) \, dt$$

$$= x_0^2 \Gamma(3) = 2x_0^2,$$

and so

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = x_0^2,$$

and hence the standard deviation is given by

$$\sigma = x_0.$$

(b) We Fourier transform $P(x)$:

$$g(k) = \frac{1}{x_0} \int_0^\infty \exp[x(ik - 1/x_0)] \, dk,$$

$$= \frac{1}{x_0} \left[ \exp[x(ik - 1/x_0)] \right]_0^\infty$$

$$= \frac{1}{x_0} \frac{1}{1/x_0 - ik} = \frac{1}{1 - i/kx_0}.$$

The distribution of the sum of $N$ random variables is the inverse Fourier transform of $G_N(k) = g^N(k)$, i.e.

$$P_N(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikX}}{(1 - i/kx_0)^N} \, dk = \frac{1}{2\pi} \frac{1}{(-ix_0)^N} \int_{-\infty}^{\infty} \frac{e^{-ikX}}{(k + i/x_0)^N} \, dk.$$

The integral has an $N$-th order pole at $k = -i/X$, at which the residue is (see the chapter in the book on contour integration)

$$\frac{1}{(N-1)!} \frac{d^{N-1}}{dk^{N-1}} \left[ e^{-ikX} \right]_{k=-i/X} = \frac{(-iX)^{N-1} e^{-X/x_0}}{(N-1)!}.$$

For $X < 0$ we have to complete the contour in the upper half plane (the contribution from the semicircle then gives zero according to Jordan’s Lemma, as discussed in the class handout “Some comments on contour integrals”). Since there are no poles inside, the integral is zero. If $X > 0$ then we complete in the lower half plane where we pick up the pole contribution, with a minus sign since we go round clockwise. Hence we have

$$P_N(X) = 0, \quad (X < 0),$$

while for $X > 0$,

$$P_N(X) = -\frac{1}{2\pi} \frac{2\pi i}{(-ix_0)^N} \frac{(-iX)^{N-1} e^{-X/x_0}}{(N-1)!}$$

$$= \frac{1}{(N-1)!} \frac{X^{N-1}}{x_0^N} \exp(-X/x_0), \quad (X \geq 0).$$
(c) The mean of $X$ is given by

$$\mu_X \equiv \langle X \rangle = \frac{1}{(N-1)!} \int_0^\infty X X^{N-1} \exp[-X/x_0] dX,$$

$$= \frac{1}{(N-1)!} x_0 \int_0^\infty t t^{N-1} \exp[-t] dt,$$

$$= \frac{N!}{(N-1)!} x_0 = N x_0 = \left[ N \mu \right] \quad (1)$$

where again we used results for $\Gamma$ functions. Similarly

$$\langle X^2 \rangle = \frac{1}{(N-1)!} \int_0^\infty X^2 X^{N-1} \exp[-X/x_0] dX,$$

$$= \frac{1}{(N-1)!} x_0^2 \int_0^\infty t^2 t^{N-1} \exp[-t] dt,$$

$$= \frac{(N+1)!}{(N-1)!} x_0^2 = N(N+1) x_0^2,$$

and so the variance of the sum is given by

$$\sigma_X^2 \equiv \langle X^2 \rangle - \langle X \rangle^2 = N \sigma^2. \quad (2)$$

Note: Equations (1) and (2) are general results for sums of independent random variables.

5. The best estimate for the true mean $\mu$ is the sample mean: $(1/2)(1 + 3) = 2$. To get the error in this we first determine the standard deviation of the sample $\sigma_{\text{samp}} = \sqrt{(1/2)(1^2 + 1^2)} = 1$. We then divide by $\sqrt{N-1}$, where $N$ is the number of data points (2 here). Hence the error bar in the mean is $1$. To summarize we obtain

$$\mu = 2 \pm 1.$$

6. The best estimate for $\langle x \rangle$ is the sample mean, given by

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i = \frac{1}{5} (1.1 + 0.9 + 0.95 + 1.05 + 1.0) = 1.0,$$

since there are $N = 5$ data points. The sample variance is given by

$$\sigma_{\text{samp}}^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2 = \frac{1}{5} \left( (0.1)^2 + (-0.1)^2 + (-0.05)^2 + (0.05)^2 + 0^2 \right) = 0.005.$$

The error bar in the mean is given by

$$\sigma_{\text{mean}} = \frac{\sigma_{\text{samp}}}{\sqrt{N-1}} = \frac{\sqrt{0.005}}{2} = \frac{1}{2 \sqrt{2}} = 0.035 \ldots.$$