PHYSICS 116C
Homework 4
Due in class, Thursday October 24

1. Note: This question is longer than the others and will get more points. It is important that you make an effort to answer it fully, since it summarizes much of what we have been doing for the last two weeks.

Consider the following differential equations which we have been discussing in class:

- The simple harmonic oscillator \((n \text{ integer so the solutions have periodicity } 2\pi)\)
  \[ y'' + n^2 y = 0 \quad (0 \leq x \leq 2\pi), \quad (1) \]

- Legendre’s equation \((n \text{ integer, otherwise one can show that } y \to \infty \text{ for } x = \pm 1)\)
  \[ (1 - x^2)y'' - 2xy' + n(n + 1)y(x) = 0 \quad (-1 \leq x \leq 1), \quad (2) \]

- Bessel’s equation \((c_n \text{ a zero of } J_{\nu}(x) \text{ so } y(1) = 0)\)
  \[ x^2y'' + xy' + (c_n^2 x^2 - \nu^2)y = 0, \quad (0 \leq x \leq 1, \quad \nu \text{ fixed}), \quad (3) \]

- Hermite’s equation \((n \text{ integer, otherwise one can show that } y \to \infty \text{ for } x \to \pm \infty)\)
  \[ y'' - 2xy' + 2ny = 0 \quad (-\infty < x < \infty), \quad (4) \]

- Laguerre’s equation \((n \text{ integer, otherwise one can show that } y \to \infty \text{ for } x \to \infty)\)
  \[ xy'' + (1 - x)y' + ny = 0 \quad (0 \leq x < \infty). \quad (5) \]

Each of these equations involves a parameter which can take more than one value, set by the boundary conditions (indicated) at the end points, and which we will generically call \(\lambda\). These discrete values of \(\lambda\) which we write as \(\lambda_n\), are known as eigenvalues. There is an analogy between these eigenvalues (of a differential equation) and the eigenvalues of matrix (which you have already met).

(a) Show that each of these equations can be written in the following form:

\[
\frac{d}{dx}\left[A(x)y'\right] + \left[\lambda w(x) + B(x)\right] y = 0, \quad (6)
\]

called the Sturm-Liouville equation, in which \(A(x), w(x)\) and \(B(x)\) are, in general, functions of \(x\). You should determine \(\lambda, A(x), w(x)\) and \(B(x)\) for each case.

Note: For some of the equations you may need to multiply by an appropriate function \(f(x)\) to cast it in the Sturm-Liouville form.

(b) Using the method discussed in class, show that if \(y_1\) and \(y_2\) are solutions of Eq. (6) corresponding to (distinct) values \(\lambda_1\) and \(\lambda_2\) of the parameter \(\lambda\), then

\[
\int_a^b w(x)y_1(x)y_2(x) \, dx = 0, \quad (7)
\]

provided

\[
\left[A(x)(y_1'y_2 - y_2'y_1)\right]^b_a = 0, \quad (8)
\]

i.e. \(y_1\) and \(y_2\) are orthogonal in the interval from \(a\) to \(b\) with “weight factor” \(w(x)\).
(c) For each of the equations:
   
i. Show that Eq. (8) is satisfied. \(a\) and \(b\) are given by the ends of the range of \(x\) specified in the question.
   
ii. Show that Eq. (7) is the orthogonality relation discussed or stated in class.

(d) Assuming “completeness” of the set of functions \(y_n(x)\), i.e. assuming that a function \(f(x)\) defined in the interval from \(a\) to \(b\) can be written as the series,

\[
f(x) = \sum_{n} a_n y_n(x),
\]

show that the coefficients \(a_n\) are given by

\[
a_n = \frac{1}{I_n} \int_{a}^{b} w(x) y_n(x) f(x) \, dx,
\]

where the “normalization” integral \(I_n\) is given by

\[
I_n = \int_{a}^{b} w(x) y_n^2(x) \, dx.
\]

(e) For the solutions of Eq. (1), show that Eqs. (9) and (10) give the familiar Fourier series.

2. (a) Use the divergence theorem

\[
\iiint_{R} \nabla \cdot \mathbf{F} \, dV = \iint_{B} \mathbf{F} \cdot d\mathbf{S}
\]

to show that

\[
\iiint_{R} (u \nabla^2 u + |\nabla u|^2) \, dV = \iint_{B} u \frac{\partial u}{\partial n} \, dS,
\]

where \(R\) is a three-dimensional region and \(B\) is the boundary of that region.

*Hint*: Choose \(\mathbf{F} = u \nabla u\).

(b) Use Eq. (11) to show that the solution of Laplace’s equation in a region is unique if its value is given everywhere on the boundary.

*Note*: we gave a different proof of this in class.

(c) Use Eq. (11) to show that the solution of Laplace’s equation in a region is uniquely determined, up to an additive constant, if the value of the normal derivative, \(\partial u/\partial n\) is given everywhere on the boundary.

3. The diffusion equation is

\[
\frac{\partial u}{\partial t} = D \nabla^2 u,
\]

where \(D\) is the diffusion constant.

(a) Verify that

\[
u(x, t) = \frac{1}{t^{1/2}} \exp\left(-\frac{x^2}{4Dt}\right)
\]

is a solution to the one-dimensional diffusion equation.

*Note*: Verify means substitute the given solution into the equation and check that it works.
(b) Verify that
\[ u(\vec{r}, t) = \frac{1}{t^{3/2}} \exp \left( -\frac{r^2}{4Dt} \right) \]
is a solution to the diffusion equation in spherical polar coordinates.

4. Verify that
\[ u(x, y, t) = \sin(k_x x) \sin(k_y y) \sin(kvt), \]
with \( k^2 = k_x^2 + k_y^2 \), is a solution to the two-dimensional wave equation. (Here \( v \) is the speed of the wave.)

5. Find the solution of Laplace’s equation in the rectangular region \( 0 \leq x \leq a, 0 \leq y \leq b \) which satisfies the boundary conditions
\[ u(0, y) = u(a, y) = u(x, 0) = 0, \quad \text{and} \quad u(x, b) = u_0 \sin(\pi x/a). \]

6. Find the solution of Laplace’s equation in the rectangular region \( 0 \leq x \leq a, 0 \leq y \leq b \) which satisfies the boundary conditions
\[ u(0, y) = u(a, y) = u(x, 0) = 0, \quad \text{and} \quad u(x, b) = u_0 x(a - x). \]

7. In class we derived the following expression
\[ T(x, y) = 4T_0 \sum_{n=1}^{\infty} \frac{\sin\left[\frac{(2n-1)\pi x}{L}\right]}{(2n - 1)\pi} e^{-(2n-1)\pi y/L} \] (12)
for the steady-state temperature in a semi-infinite slab \( (0 \leq x \leq L, 0 \leq y < \infty) \) under the boundary conditions \( T(0, y) = T(L, y) = 0, T(x, 0) = T_0 \) and \( \lim_{y \to \infty} T(x, y) \to 0 \). Show that the series can be summed to the closed form expression
\[ T(x, y) = \frac{2T_0}{\pi} \tan^{-1} \left[ \frac{\sin(\pi x/L)}{\sinh(\pi y/L)} \right], \] (13)
and verify that Eq. (13) satisfies the boundary conditions.

Hints:
(i) Write \( \sin(m \pi x/L) = \text{Im} e^{im \pi x/L} \) and hence show that the series is of the form \( \text{Im} \sum_{\text{odd}} m \frac{z^m}{m} \) (for an appropriate choice of \( z \) which you should determine).
(ii) Compare this series with the series for \( \ln[(1 + z)/(1 - z)] \).
(iii) You then extract the imaginary part of the log.

Note:
Although the series solution, Eq. (12), consists of an infinite number of terms each of which is separable, i.e. is the product of a function of \( x \) times a function of \( y \), the final closed form solution, Eq. (13), is not separable.