PHYSICS 116A
Homework 9 Solutions

1. Boas, problem 3.12–4. Find the equations of the following conic,

$$3x^2 + 8xy - 3y^2 = 8,$$  

relative to the principal axes.

In matrix form, Eq. (1) can be written as:

$$(x \ y) \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 8.$$

I could work out the eigenvalues by solving the characteristic equation. But, in this case I can work them out by inspection by noting that for the matrix

$$M = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix},$$

we have

$$\lambda_1 + \lambda_2 = \text{Tr} \ M = 0, \quad \lambda_1 \lambda_2 = \det \ M = -25.$$

It immediately follows that the two eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -5$. Next, we compute the eigenvectors.

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix}$$

yields one independent relation, $x = 2y$. Thus, the normalized eigenvector is

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=5} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Since $M$ is a real symmetric matrix, the two eigenvectors are orthogonal. It follows that the second normalized eigenvector is:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=-5} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

The two eigenvectors form the columns of the diagonalizing matrix,

$$C = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$  \hspace{1cm} (2)

Since the eigenvectors making up the columns of $C$ are real orthonormal vectors, it follows that $C$ is a real orthogonal matrix, which satisfies $C^{-1} = C^T$. As a check, we make sure that $C^{-1}MC$ is diagonal.

$$C^{-1}MC = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 10 & 5 \\ 5 & -10 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}.$$
Following eq. (12.3) on p. 162 of Boas, 
\[
\begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} x' \\ y' \end{pmatrix},
\]
where \((x', y')\) are the principal axes. Hence, using \(C^T = C^{-1}\), it follows that:
\[
\begin{pmatrix} x \\ y \end{pmatrix} M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} C^T MC \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} C^{-1} MC \begin{pmatrix} x' \\ y' \end{pmatrix} = 5(x'^2 - y'^2),
\]
That is, relative to the principal axes, Eq. (1) takes the form
\[
5(x'^2 - y'^2) = 8
\]

2. Boas, problem 3.12–16. Find the characteristic frequencies and the characteristic modes of vibration for the system of masses and springs as in Figure 12.1 on p. 165 of Boas, for the following array: \(4k, m, 2k, m, k\).

Following the discussion of Example 3 on pp. 165–166 of Boas, the total potential energy is
\[
V = \frac{1}{2}(4k)x^2 + \frac{1}{2}(2k)(x - y)^2 + \frac{1}{2}ky^2 = k(3x^2 - 2xy + \frac{3}{2}y^2).
\]
The equations of motion are:
\[
\begin{align*}
\begin{cases}
  m\ddot{x} = -\partial V/\partial x = -6kx + 2ky, \\
  m\ddot{y} = -\partial V/\partial y = 2kx - 3ky,
\end{cases}
\end{align*}
\]
where \(\ddot{x} \equiv d^2x/dt^2\) and \(\ddot{y} \equiv d^2y/dt^2\). Assuming solutions of the form \(x = x_0e^{i\omega t}\) and \(y = y_0e^{i\omega t}\), it follows that
\[
\ddot{x} = -\omega^2x \quad \text{and} \quad \ddot{y} = -\omega^2y.
\]
Substituting these results into Eq. (4) yields:
\[
\begin{align*}
\begin{cases}
  -m\omega^2x = -\partial V/\partial x = -6kx + 2ky, \\
  -m\omega^2y = -\partial V/\partial y = 2kx - 3ky,
\end{cases}
\end{align*}
\]
In matrix form, these equations are:
\[
\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{with} \quad \lambda \equiv \frac{m\omega^2}{k}. \quad (5)
\]
But the coefficients matrix above is precisely the matrix whose eigenvalue problem was solved in Qu. (10) on homework set 8. Thus, the eigenvalues are \(\lambda = 7\) and \(\lambda = 2\) with corresponding normalized eigenvectors
\[
\begin{align*}
\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=7} &= \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\end{align*}
\]
Using Eq. (5), \( \omega = (\lambda k/m)^{1/2} \). Thus, there are two characteristic frequencies,

\[
\omega_1 = \sqrt{\frac{7k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{2k}{m}}.
\]

The corresponding characteristic modes of vibrations are identified by the unnormalized eigenvectors,

\[
\begin{pmatrix} x \\ y \end{pmatrix}_{\omega_1} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix}_{\omega_2} = c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix},
\]

where \( c_1 \) and \( c_2 \) are non-zero constants. These modes describe displacements which oscillate at a single frequency. The general solution to the equations of motion is a linear combination of these two modes and so exhibits oscillations at two frequencies.

3. Boas, problem 10.2–7. Following eqs. (2.14)–(2.17) on pp. 500–501 of Boas, show that the direct product of two tensors of ranks \( m \) and \( n \) is a fourth-rank tensor. Generalize these results to show that the direct product of two tensors of ranks \( m \) and \( n \) is a tensor of rank \( m + n \).

Here, I follow the notation of Boas, and use \( a_{ij} \) for the matrix elements of the rotation matrix (rather than \( R_{ij} \) which was used in class).

An \( n \)th rank Cartesian tensor is defined according to the transformation law of its components with respect to rotations (either proper or improper). In particular,

\[
T'_{1_1 2_1 3_1 \ldots i_n} = a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n} T_{1_1 2_1 3_1 \ldots i_n},
\]

(6)

where the Einstein summation convention is being used to sum over the repeated indices \( j_1, j_2, j_3, \ldots, j_n \). Thus, if I multiply an \( n \)th rank Cartesian tensor by an \( m \)th rank Cartesian tensor, where the transformation property of the latter is

\[
S'_{k_1 k_2 k_3 \ldots k_m} = a_{k_1 \ell_1} a_{k_2 \ell_2} a_{k_3 \ell_3} \cdots a_{k_m \ell_m} S_{\ell_1 \ell_2 \ell_3 \ldots \ell_m},
\]

(7)

then one obtains a new tensor with \( n + m \) indices:

\[
P'_{1_1 2_1 3_1 \ldots i_n k_1 k_2 k_3 \ldots k_m} = T_{1_1 2_1 3_1 \ldots i_n} S_{k_1 k_2 k_3 \ldots k_m}.
\]

This is called a direct product of two tensors. Relative to the rotated coordinate system,

\[
P'_{1_1 2_1 3_1 \ldots i_n k_1 k_2 k_3 \ldots k_m} = T'_{1_1 2_1 3_1 \ldots i_n} S'_{k_1 k_2 k_3 \ldots k_m}.
\]

Hence, the transformation law for the product tensor \( P \) follows immediately from Eqs. (6) and (7):

\[
P'_{1_1 2_1 3_1 \ldots i_n k_1 k_2 k_3 \ldots k_m} = (a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} \cdots a_{i_n j_n})(a_{k_1 \ell_1} a_{k_2 \ell_2} a_{k_3 \ell_3} \cdots a_{k_m \ell_m}) P_{j_1 j_2 j_3 \cdots j_n \ell_1 \ell_2 \ell_3 \ldots \ell_m},
\]

which is the correct transformation law for a tensor of rank \( m + n \).

Specializing the above proof to the direct product of a vector and a third-rank tensor and to the direct product of two second-rank tensors is immediate, and I will not write out the proof explicitly.
4. Boas, problem 10.3–3. Show that the contracted tensor \( T_{iik} \) is a first-rank tensor, that is, a vector.

The transformation law of a third-rank tensor \( T_{ijk} \) due to the rotation of the coordinate system is given by:

\[
T'_{ijk} = a_{i\ell}a_{jm}a_{kn}T_{\ell mn},
\]  

(8)

where I am using the Einstein summation convention to sum implicitly over the repeated indices \( \ell, m \) and \( n \). Now, set \( i = j \) in Eq. (8) and sum over \( j \). One obtains

\[
T'_{iik} = a_{i\ell}a_{im}a_{kn}T_{\ell mn},
\]  

(9)

Recall that \( A = [a_{ij}] \) is an orthogonal matrix, which means that \( A^TA = I \). In component form we have:

\[
(A^TA)_{jk} = a_{ij}a_{ik} = \delta_{jk}.
\]  

(10)

Applying Eq. (10) in Eq. (9),

\[
T'_{iik} = \delta_{\ell m}a_{kn}T_{\ell mn} = a_{kn}T_{mmn},
\]  

(11)

since the effect of multiplying by the Kronecker delta is to simply set \( \ell = m \) in the remaining terms. If we define \( v_k \equiv T_{iik} \) (where there is an implicit sum over \( i \) in \( T_{iik} \)), then, Eq. (11) yields

\[
v'_k = a_{kn}v_n,
\]

which is the correct transformation law for a first-rank tensor, that is, a vector. Hence we have demonstrated that \( T_{iik} \) transforms like a vector.

5. Boas, problem 10.4–5. Given two point masses of equal mass \( m = 1 \) located at the coordinates \((1, 1, 1)\) and \((-1, 1, 1)\), find the inertia tensor about the origin, and find the principal moments and the principal axes.

The moment of inertia tensor, with respect to the origin, for point masses is given by [cf. eq. (4.5) on p.506 of Boas and eq. (7.8) on p. 519 of Boas]:

\[
I_{ij} = \sum_a m_a \left( \delta_{ij} r_a^2 - x_{ai} x_{aj} \right),
\]  

(12)

where \( a \) labels the mass points, each of which is located at position \((x_{a1}, x_{a2}, x_{a3})\), and \( r^2 \equiv x_{a1}^2 + x_{a2}^2 + x_{a3}^2 \). Inserting the coordinates \((1,1,1)\) and \((-1,1,1)\) into Eq. (12) yields the following matrix representation of \( I_{ij} \),

\[
I_{ij} = \begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & -2 \\
0 & -2 & 4
\end{pmatrix}.
\]

The principal moments are the eigenvalues of \( I_{ij} \), which are determined by solving the characteristic equation,

\[
\begin{vmatrix}
4 - \lambda & 0 & 0 \\
0 & 4 - \lambda & -2 \\
0 & -2 & 4 - \lambda
\end{vmatrix} = (4-\lambda) \left[ (4-\lambda)^2 - 4 \right] = (4-\lambda) \left[ \lambda^2 - 8\lambda + 12 \right] = (2-\lambda)(4-\lambda)(6-\lambda) = 0.
\]
Thus, the principal moments are 2, 4 and 6.

The principal axes correspond to the eigenvectors of $I_{ij}$. Corresponding to $\lambda = 2$,

$$
\begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & -2 \\
0 & -2 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 2
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix},
$$

which implies $x = 0$ and $y = z$. Corresponding to $\lambda = 4$,

$$
\begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & -2 \\
0 & -2 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 4
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix},
$$

which implies $y = z = 0$. Corresponding to $\lambda = 6$,

$$
\begin{pmatrix}
4 & 0 & 0 \\
0 & 4 & -2 \\
0 & -2 & 4
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 6
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix},
$$

which implies $x = 0$ and $y = -z$. Normalizing the eigenvectors, the above results yield the following principal axes corresponding to $\lambda = 2$, 4 and 6, respectively,

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.
$$

6. Show that a triple scalar product $\vec{a} \cdot \vec{b} \times \vec{c}$ is a pseudoscalar. Assume that $\vec{a}, \vec{b}$ and $\vec{c}$ are ordinary (i.e. polar) vectors. Hint: The transformation of the elements of Levi-Civita tensor is discussed in appendix D of the handout on tensors.

We have

$$
\vec{a} \cdot \vec{b} \times \vec{c} = \epsilon_{ijk} a_i b_j c_k. \quad (13)
$$

In appendix D it is shown that $\epsilon$ is a third-rank tensor under rotations. Since all the indices in Eq. (13) are contracted, we conclude that the triple scalar product is indeed a scalar under rotations.

However, under improper rotations, appendix D shows that $\epsilon$ is a pseudo-tensor. Since we are told that $\vec{a}, \vec{b}$ and $\vec{c}$ are polar vectors, it follows that the triple scalar product in Eq. (13) is a pseudo-scalar. Indeed, by considering the special case of inversion, we see that the triple scalar product changes sign, whereas a true scalar would not.

7. The elastic constants of a material form a fourth rank Cartesian tensor, $C_{ijkl}$.

(a) Explain why, if there were no relations between the different elements, there would be 81 elements. (Easy).

Each of the four indices runs over three possible values so the total number is $3^4 = 81$. 

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(b) In fact the elastic constant tensor is invariant under interchange of the first two indices or the
last two indices, i.e. $C_{ijkl} = C_{jikl} = C_{ijlk}$. Show that this reduces the number of independent
components to 36.

The number of possible distinct values of the elastic constant coming from the pair $i$ and $j$,
given that $i j$ is the same as $j i$ is (a) 3 (where $i = j$) plus (b) $3 \cdot 2/2 = 3$ (where $i \neq j$). So
the total is 6. Similarly the number of possibilities coming from the pair $k$ and $l$ is 6. Hence
the total number is $6^2 = 36$.

(c) In addition the elastic constant tensor is invariant under interchange of the first pair of indices
with the second pair, i.e. $C_{ijkl} = C_{klij}$. Show that this reduces the number of independent
components further to 21.

There are 6 choices for the pair $ij$ and 6 for the pair $kl$. The number of choices where the
pair $ij$ is the same as the pair $kl$ is 6. The number of choices where the pair $ij$ is distinct
from that of the pair $kl$ (irrespective of which came first) is $6 \times 5/2 = 15$. Hence the total
number is $6 + 15 = 21$.

Note: The actual number of independent components may be much less than this depending on
the symmetry of the system.

8. (a) Show that

$$\epsilon_{ikl} \epsilon_{jkl} = 2 \delta_{ij},$$

where $\epsilon_{ijk}$ is the completely antisymmetric isotropic third rank tensor discussed in class.

We need to determine $\epsilon_{ikl} \epsilon_{jkl}$ ($k$ and $l$ summed over). Suppose $i \neq j$, e.g. $i = 1$, $j = 2$, so we
need $\epsilon_{1kl} \epsilon_{2kl}$. If any two indices on $\epsilon$ are equal then $\epsilon = 0$. Hence for $\epsilon_{1kl}$ we need $k = 2$, $l = 3$
(or $k = 3$, $l = 2$) but then $\epsilon_{2kl} = 0$ because two of its indices are equal to 2. Hence

$$\epsilon_{ikl} \epsilon_{jkl} = 0 \text{ if } i \neq j.$$

Now assume that $i = j$ (= 1 say). Then we have a contribution when $k = 2$, $l = 3$ and
$k = 3$, $l = 2$ (other possibilities for $k$, $l$ give zero because of repeated indices). Hence

$$\epsilon_{1kl} \epsilon_{1kl} = \epsilon_{123}^2 \quad (k = 2, l = 3)$$
+ $\epsilon_{132}^2 \quad (k = 3, l = 2)$

$$= 1 + 1 = 2.$$  

Similarly for $i = j = 2$ or 3. Hence we have

$$\epsilon_{ikl} \epsilon_{jkl} = 2 \delta_{ij}.$$  

(b) By considering the various possibilities for the indices $j, k, l, m$ show that

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl},$$

which was discussed in class.

We need to show that

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}.$$  

(14)
The RHS is +1 if \( j = l, k = m \) \((j \neq k)\), is \(-1\) if \( j = m, k = l \) \((j \neq k)\), and is 0 otherwise. Suppose that \( j = l = 2, k = m = 3 \). The LHS of Eq. (14) is
\[
\epsilon_{123}\epsilon_{123} + \epsilon_{223}\epsilon_{223} + \epsilon_{323}\epsilon_{323} = 1 + 0 + 0 = 1,
\]
which is equal to the RHS.
Suppose that \( j = m = 2, k = l = 3 \). The LHS of Eq. (14) is
\[
\epsilon_{123}\epsilon_{132} + \epsilon_{223}\epsilon_{232} + \epsilon_{323}\epsilon_{332} = -1 + 0 + 0 = -1,
\]
which is equal to the RHS.
If \( j = k \) and \( l = m \) then both sides of Eq. (14) vanish.
If \( j, k, l, m \) involve all three indices 1, 2 and 3, e.g. \( j = l = 2, k = 3, m = 1 \) then the LHS of Eq. (14) is
\[
\epsilon_{123}\epsilon_{121} + \epsilon_{223}\epsilon_{221} + \epsilon_{323}\epsilon_{321} = 0 + 0 + 0 = 0,
\]
which is equal to the RHS.
This has covered all the possibilities and so Eq. (14) is valid.

9. State which of the following equations are sensible and explain your reasons:

(a) \( A_{ij}x_j = B_i \)

Sensible; unsummed indices on the two sides of the equations match.

(b) \( A_{ij}x_j = C \)

Not sensible. The LHS is a vector and the RHS is a scalar.

(c) \( D_{ijkl}x_ix_jx_kx_l = C \)

Sensible; unsummed indices on the two sides of the equations match.

(d) \( \epsilon_{ijk} \frac{\partial G_i}{\partial x_j} = B_k \)

Sensible; unsummed indices on the two sides of the equations match.

(e) \( \frac{\partial F_{ijk}}{\partial x_k} = B_j \)

Not sensible. The LHS is a second rank tensor and the RHS is a vector.

10. (a) Consider the contravariant four-vector, \( x^\mu, \mu = 0, 1, 2, 3 \) where \( x^0 = ct, x^1 = x, x^2 = y, x^3 = z \). Write down the corresponding covariant four-vector, \( x_\mu \).

The contravariant 4-vector is
\[
x^\mu = \begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]
The corresponding covariant 4-vector is

\[ x_\mu = \begin{pmatrix} -ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \]

(b) Which of the following is a scalar: \( x_\mu x_\mu \), \( x_\mu x_\mu \), \( x^\mu x^\mu \)?

To get a scalar we need to contract a covariant vector with a contravariant vector, so \( x^\mu x_\mu \) is a scalar while \( x_\mu x_\mu \) and \( x^\mu x^\mu \) are not scalars.

Note:

\[ x^\mu x_\mu = x_1^2 + x_2^2 + x_3^2 - (ct)^2 = r^2 - (ct)^2. \]

(c) We showed in the handout that \( \partial \phi / \partial x_\mu \) (where \( \phi \) is a scalar field) transforms like a contravariant four-vector, and hence is often written \( \partial^\mu \phi \). Hence show that the operator

\[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \]

is a scalar.

\[ \frac{\partial}{\partial x_\mu} \equiv \partial^\mu \quad \text{is contravariant}. \]

\[ \frac{\partial}{\partial x^\mu} \equiv \partial_\mu \quad \text{is covariant}. \]

Hence \( \partial^\mu \partial_\mu \) is a scalar operator. Now

\[ \partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c} \frac{\partial}{\partial t}, \quad \partial^0 = \frac{\partial}{\partial x_0} = -\frac{1}{c} \frac{\partial}{\partial t}, \quad \partial_i = \partial^i = \frac{\partial}{\partial x_i} (i = 1, 2, 3). \]

Hence

\[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \partial^\mu \partial_\mu \]

which is a scalar.