PHYSICS 116A
Homework 8 Solutions

1. Boas, problem 3.9–4. Given the matrix,

\[ A = \begin{pmatrix} 0 & 2i & -1 \\ -i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}, \]

find the transpose, the inverse, the complex conjugate and the transpose conjugate of \( A \). Verify that \( AA^{-1} = A^{-1}A = I \), where \( I \) is the identity matrix.

We shall evaluate \( A^{-1} \) by employing Eq. (6.13) in Ch. 3 of Boas. First we compute the determinant by expanding in cofactors about the third column:

\[
\det A = \begin{vmatrix} 0 & 2i & -1 \\ -i & 2 & 0 \\ 3 & 0 & 0 \end{vmatrix} = (-1) \begin{vmatrix} -i & 2 \\ 3 & 0 \end{vmatrix} = (-1)(-6) = 6.
\]

Next, we evaluate the matrix of cofactors and take the transpose:

\[
\text{adj } A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & i \\ -6 & 6i & -2 \end{pmatrix}.
\]

According to Eq. (6.13) in Ch. 3 of Boas the inverse of \( A \) is given by Eq. (1) divided by \( \det(A) \), so

\[
A^{-1} = \begin{pmatrix} 0 & \frac{1}{5} \\ 0 & \frac{i}{5} \\ -1 & -\frac{i}{5} \end{pmatrix}.
\]

It is straightforward to check by matrix multiplication that

\[
\begin{pmatrix} 0 & 2i & -1 \\ -i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{5} \\ 0 & \frac{i}{5} \\ -1 & -\frac{i}{5} \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{5} \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The transpose, complex conjugate and the transpose conjugate* can be written down by inspection:

\[
A^T = \begin{pmatrix} 0 & -i & 3 \\ 2i & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & -2i & -1 \\ i & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} 0 & i & 3 \\ -2i & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}.
\]

*The transpose conjugate is more often referred to as the hermitian conjugate or the adjoint.
These have been obtained by noting that \( A^T \) is obtained by A by interchanging the rows and columns, \( A^* \) is obtained from \( A \) by complex conjugating the matrix elements, and the definition of the hermitian conjugate is \( A^\dagger = (A^*)^T \).

**Alternative method:** One can also evaluate \( A^{-1} \) by employing Gauss Jordan elimination, which is described in the class handout [http://young.physics.ucsc.edu/116A/gauss_jordan.pdf](http://young.physics.ucsc.edu/116A/gauss_jordan.pdf).

\[
A = \begin{pmatrix}
0 & 2i & -1 \\
-i & 2 & 0 \\
3 & 0 & 0
\end{pmatrix}, \quad I = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

First we interchange \( R_1 \leftrightarrow R_3 \). Then we rescale the new row 1, \( R_1 \rightarrow \frac{1}{3} R_1 \) to obtain

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2i & -1 \\
-i & 2 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & \frac{1}{3} \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

Next, we perform the elementary row operations, \( R_3 \rightarrow R_3 + iR_1 \) and \( R_2 \rightarrow -\frac{1}{2}iR_2 \) to obtain:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{2}i \\
0 & 2 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & \frac{1}{3} \\
-\frac{1}{2}i & 0 & 0 \\
0 & 1 & \frac{1}{3}i
\end{pmatrix}
\]

Next, we perform the elementary row operations, \( R_3 \rightarrow R_3 - 2R_1 \) to obtain:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{2}i \\
0 & 0 & -i
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & \frac{1}{3} \\
-\frac{1}{2}i & 0 & 0 \\
i & 1 & \frac{1}{3}i
\end{pmatrix}
\]

Finally, we perform the elementary row operations, \( R_2 \rightarrow R_2 + \frac{1}{2}R_3 \), followed by \( R_3 \rightarrow iR_3 \) to obtain:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad A^{-1} = \begin{pmatrix}
0 & 0 & \frac{1}{3} \\
0 & \frac{1}{2} & \frac{i}{6} \\
-1 & i & \frac{-1}{3}
\end{pmatrix}
\]

The final step produces the inverse, which is indicated above. Note that we have reproduced Eq. (2).

2. Boas, problem 3.9–5. Show that the product \( AA^T \) is a symmetric matrix.

Using Eq. (9.10) on p. 139 of Boas, \((AB)^T = B^T A^T\) for any two matrices \( A \) and \( B \). Hence,

\[
(AA^T)^T = (A^T)^T A^T = [AA^T],
\]

where we have used the fact\(^\dagger\) that \((A^T)^T = A\) for any matrix \( A \). Eq. (3) implies that \( AA^T \) is a symmetric matrix, since by definition a symmetric matrix is equal to its transpose [cf. the table at the top of p. 138 of Boas].

\(^\dagger\)The transpose of a matrix interchanges the rows and columns. Thus, if one performs the transpose operation twice, the original matrix is recovered.

(a) Show that if \( A \) and \( B \) are symmetric, then \( AB \) is not symmetric unless \( A \) and \( B \) commute.

If \( A \) and \( B \) are symmetric, then \( A = A^T \) and \( B = B^T \). We now examine
\[
(AB)^T = B^T A^T = BA,
\]
after using the fact that \( A \) and \( B \) are symmetric matrices. We conclude that \( (AB)^T = AB \) if and only if \( AB = BA \). That is, \( AB \) is not symmetric unless \( A \) and \( B \) commute.

(b) Show that a product of orthogonal matrices is orthogonal.

Consider orthogonal matrices \( Q_1 \) and \( Q_2 \). By definition [cf. the table at the top of p. 138 of Boas], we have \( Q_1^{-1} = Q_1^T \) and \( Q_2^{-1} = Q_2^T \). We now compute
\[
(Q_1 Q_2)^{-1} = Q_2^{-1} Q_1^{-1} = Q_2^T Q_1^T = (Q_1 Q_2)^T,
\]
after using the fact that \( Q_1 \) and \( Q_2 \) are orthogonal. In deriving Eq. (4), we have used the following properties of the inverse and the transpose
\[
(AB)^{-1} = B^{-1} A^{-1}, \quad \text{and} \quad (AB)^T = B^T A^T,
\]
for any pair of matrices \( A \) and \( B \). Thus, we have shown that
\[
(Q_1 Q_2)^{-1} = (Q_1 Q_2)^T,
\]
which implies that \( Q_1 Q_2 \) is orthogonal.

(c) Show that if \( A \) and \( B \) are Hermitian, then \( AB \) is not Hermitian unless \( A \) and \( B \) commute.

If \( A \) and \( B \) are Hermitian, then \( A = A^\dagger \) and \( B = B^\dagger \). We now examine
\[
(AB)^\dagger = B^\dagger A^\dagger = BA,
\]
after using the fact that \( A \) and \( B \) are Hermitian matrices. In deriving Eq. (5), we have used the fact that:
\[
(AB)^\dagger = ((AB)^*)^T = (A^* B^*)^T = (B^*)^T (A^*)^T = B^\dagger A^\dagger.
\]
We conclude that \( (AB)^\dagger = AB \) if and only if \( AB = BA \). That is, \( AB \) is not Hermitian unless \( A \) and \( B \) commute.

(d) Show that a product of unitary matrices is unitary.

Consider unitary matrices \( U_1 \) and \( U_2 \). By definition [cf. the table at the top of p. 138 of Boas], we have \( U_1^{-1} = U_1^\dagger \) and \( U_2^{-1} = U_2^\dagger \). We now compute
\[
(U_1 U_2)^{-1} = U_2^{-1} U_1^{-1} = U_2^\dagger U_1^\dagger = (U_1 U_2)^\dagger,
\]
after using the fact that \( U_1 \) and \( U_2 \) are unitary and employing the property of the Hermitian conjugation given in Eq. (6). Thus, we have shown that \( (U_1 U_2)^{-1} = (U_1 U_2)^\dagger \), which implies that \( U_1 U_2 \) is orthogonal.
4. Boas, problem 3.10-5(a). Given two vectors,
\[ \vec{A} = (3 + i, 1, 2 - i, -5i, i + 1) \quad \text{and} \quad \vec{B} = (2i, 4 - 3i, 1 + i, 3i, 1), \]
find the norms of \( \vec{A} \) and \( \vec{B} \) and the inner product of \( \vec{A} \) and \( \vec{B} \), and note that the Schwarz inequality is satisfied.

Using eq. (1.07) on p. 146 of Boas, the norms of \( \vec{A} \) and \( \vec{B} \) are given by:
\[
\|\vec{A}\| = (|3 + i|^2 + |1|^2 + |2 - i|^2 + |-5i|^2 + |i + 1|^2)^{1/2} = (10 + 1 + 5 + 25 + 2)^{1/2} = \sqrt{43},
\]
\[
\|\vec{B}\| = (|2i|^2 + |4 - 3i|^2 + |1 + i|^2 + |3i|^2 + 1)^{1/2} = (4 + 25 + 2 + 9 + 1)^{1/2} = \sqrt{41}.
\]

Using eq. (10.6) on p. 146 of Boas, the inner product of \( \vec{A} \) and \( \vec{B} \) is given by:
\[
\vec{A} \cdot \vec{B} = (3 - i)(2i) + (1)(4 - 3i) + (2 + i)(1 + i) + (5i)(3i) + (1 - i)(1)
= (2 + 6i) + (4 - 3i) + (1 + 3i) - 15 + (1 - i)
= -7 + 5i. \tag{7}
\]

The Schwarz inequality [see eq. (10.9) on p. 146 of Boas] is:
\[
|\vec{A} \cdot \vec{B}| \leq \|\vec{A}\| \|\vec{B}\|,
\]
where \(|\cdots|\) indicates the magnitude of the corresponding complex number. Using Eq. (7), it follows that
\[
|\vec{A} \cdot \vec{B}| = |-7 + 5i| = \sqrt{49 + 25} = \sqrt{74}.
\]
Thus, the Schwarz inequality is satisfied for this example since
\[
\sqrt{74} \leq \sqrt{43} \sqrt{41} = \sqrt{1763}.
\]

5. Boas, problem 3.11–9. Show that \( \det(C^{-1}MC) = \det M \). What is the product of \( \det(C^{-1}) \) and \( \det C \)? Thus, show that the product of the eigenvalues of \( M \) is equal to \( \det M \).

Eq. (6.6) on p. 118 of Boas states that \( \det(AB) = \det A \det B \) for any two matrices \( A \) and \( B \). It follows that
\[
\det(C^{-1})\det C = \det(C^{-1}C) = \det I = 1,
\]
where \( I \) is the identity matrix. Hence,\(^4\)
\[
\det(C^{-1}) = \frac{1}{\det C}.
\]

Using Eq. (6.6) of Boas again along with the result just obtained,
\[
\det(C^{-1}MC) = (\det(C^{-1} \det M \det C) = \frac{\det M \det C}{\det C} = \frac{\det M}{\det C}. \tag{8}
\]
\(^4\)By assumption, \( C^{-1} \) exists, in which case \( \det C \neq 0 \) and so it is permissible to divide by \( \det C \).
Finally, Boas asks you to show that the product of the eigenvalues of $M$ is equal to $\det M$. What she is expecting you to do is to use eq. (11.11) on p. 150 of Boas, which states that

$$C^{-1}MC = D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $D$ is a diagonal matrix whose diagonal elements are the eigenvalues of $M$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ above. It follows that

$$\det(C^{-1}MC) = \det D = \lambda_1 \lambda_2 \cdots \lambda_n. \quad (10)$$

The results of Eqs. (8) and (10) then imply that

$$\det M = \lambda_1 \lambda_2 \cdots \lambda_n. \quad (11)$$

Note that this proof is not completely general, since not all matrices $M$ are diagonalizable and, for these, the matrix of eigenvectors $C$ does not have an inverse since two or more eigenvectors are equal, with the result that $\det(C) = 0$. However, one can show that Eq. (11) is also true for non-diagonalizable matrices.

6. Boas, problem 3.11–10. Show that $\text{Tr}(C^{-1}MC) = \text{Tr} M$. Thus, show that the sum of the eigenvalues of $M$ is equal to $\text{Tr} M$.

Eq. (9.13) on p. 140 of Boas states that the trace of a product of matrices is not changed by permuting them in cyclic order. In particular, $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$ for any three matrices $A$, $B$ and $C$. It follows that

$$\text{Tr}(C^{-1}MC) = \text{Tr}(MCC^{-1}) = \text{Tr}(MI) = \text{Tr} M, \quad (12)$$

where $I$ is the identity matrix.

Finally, Boas asks you to show that the sum of the eigenvalues of $M$ is equal to $\text{Tr} M$. What she is expecting you to do is to use Eq. (9) to obtain

$$\text{Tr}(C^{-1}MC) = \text{Tr} D = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \quad (13)$$

where $D$ is a diagonal matrix whose diagonal elements are the eigenvalues of $M$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ above. The results of Eqs. (12) and (13) then imply that

$$\text{Tr} M = \lambda_1 + \lambda_2 + \cdots + \lambda_n. \quad (14)$$

As in the previous question, this proof is not general, since not all matrices $M$ are diagonalizable. However, one can show that Eq. (14) is also true for non-diagonalizable matrices.

7. Boas, problem 3.11–19. Find the eigenvalues and eigenvectors of

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$
To compute the eigenvalues, we evaluate the characteristic equation,

\[ \det(M - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 2 \\ 2 & 3 - \lambda & 0 \\ 2 & 0 & 3 - \lambda \end{vmatrix} = 0. \]

Expanding in terms of the cofactors of the elements of the third row,

\[ \begin{vmatrix} 1 - \lambda & 2 & 2 \\ 2 & 3 - \lambda & 0 \\ 2 & 0 & 3 - \lambda \end{vmatrix} = -4(3 - \lambda) + (3 - \lambda) [(1 - \lambda)(3 - \lambda) - 4] \]

\[ = (3 - \lambda)(1 - \lambda)(3 - \lambda) - 8 \]

\[ = (3 - \lambda)(\lambda^2 - 4\lambda - 5) \]

\[ = -\lambda(3)(\lambda - 5)(\lambda + 1). \]

Hence, the characteristic equation possesses three roots: \( \lambda = -1, 3, \text{ and } 5. \)

To obtain the eigenvectors, we plug in the eigenvalues into the equation \( M \vec{v} = \lambda \vec{v} \). First, we consider the eigenvalue \( \lambda = 5 \).

\[ \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \]

Expanding out this matrix equation yields:

\[ x + 2y + 2z = 5x, \]
\[ 2x + 3y = 5y, \]
\[ 2x + 3z = 5z. \]

Rewrite the above equations as a set of homogeneous equations,

\[ -4x + 2y + 2z = 0, \]
\[ 2x - 2y = 0, \]
\[ 2x - 2z = 0. \]

The solution can be determined by inspection,

\[ x = y = z. \]

Normalizing the eigenvector yields (up to an overall sign):

\[ \vec{v}(\lambda=5) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \]
which satisfies $M \vec{v} = \lambda \vec{v}$ with eigenvalue $\lambda = 5$.

Next, we consider the eigenvalue $\lambda = 3$.

\[
\begin{pmatrix}
1 & 2 & 2 \\
2 & 3 & 0 \\
2 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 3
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
\]

Expanding out this matrix equation yields:

\[
x + 2y + 2z = 3x,
\]
\[
2x + 3y = 3y,
\]
\[
2x + 3z = 3z.
\]

Rewrite the above equations as a set of homogeneous equations,

\[
-2x + 2y + 2z = 0,
\]
\[
2x = 0,
\]
\[
2x = 0.
\]

The solution can be determined by inspection,

\[
x = 0, \text{ and } y = -z.
\]

Normalizing the eigenvector yields (up to an overall sign):

\[
\vec{v}_{(\lambda=3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},
\]

which satisfies $M \vec{v} = \lambda \vec{v}$ with eigenvalue $\lambda = 3$.

Finally, we consider the eigenvalue $\lambda = -1$.

\[
\begin{pmatrix}
1 & 2 & 2 \\
2 & 3 & 0 \\
2 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= -
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
\]

Expanding out this matrix equation yields:

\[
x + 2y + 2z = 5x,
\]
\[
2x + 3y = 5y,
\]
\[
2x + 3z = 5z.
\]

Rewrite the above equations as a set of homogeneous equations,

\[
2x + 2y + 2z = 0,
\]
\[
2x + 4y = 0,
\]
\[
2x + 4z = 0.
\]
After subtracting the last two equations, the solution can be determined by inspection,

\[ y = z \quad \text{and} \quad x = -2y. \]

Normalizing the eigenvector yields (up to an overall sign):

\[ \vec{v}_{(\lambda=-1)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \]

which satisfies \( M \vec{v} = \lambda \vec{v} \) with eigenvalue \( \lambda = -1 \).

Note that the three eigenvectors form an orthonormal set. This is the expected behavior for a real symmetric (or complex hermitian) matrix.

8. Boas, problem 3.11–32. The matrix,

\[ M = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}, \]

describes a deformation of the \((x, y)\) plane. Find the eigenvalues and eigenvectors of the transformation, the matrix \( C \) that diagonalizes \( M \) and specifies the rotation to new axes \((x', y')\) along the eigenvectors, and the matrix \( D \) that gives the deformation relative to the new axes. Describe the deformation relative to the new axes.

First, we compute the eigenvalues by solving the characteristic equation,

\[
\begin{vmatrix}
6 - \lambda & -2 \\
-2 & 3 - \lambda
\end{vmatrix} = (6 - \lambda)(3 - \lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2) = 0.
\]

Hence, the two eigenvalues are \( \lambda = 7 \) and \( \lambda = 2 \). Next, we work out the eigenvectors. Since \( M \) is a real symmetric matrix, we know that the eigenvectors will be orthogonal. By normalizing them to unity, the eigenvectors will then be orthonormal. First, we examine:

\[
\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix},
\]

which is equivalent to

\[
6x - 2y = 2x, \\
-2x + 3y = 2y.
\]

These equations yield one independent relation, \( y = 2x \). Hence, the normalized eigenvector is

\[
\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=2} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]

The second normalized eigenvector is orthogonal to the first one, and thus is given by

\[
\begin{pmatrix} x \\ y \end{pmatrix}_{\lambda=7} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.
\]
One can check this by verifying that,
\[
\begin{pmatrix}
6 & -2 \\
-2 & 3
\end{pmatrix}
\begin{pmatrix}
2 \\
-1
\end{pmatrix}
= 7
\begin{pmatrix}
2 \\
-1
\end{pmatrix}.
\]

The columns of the diagonalizing matrix \(C\) are given by the two eigenvectors. Thus,
\[
C = \frac{1}{\sqrt{5}} \begin{pmatrix}
1 \\
2
\end{pmatrix}.
\]

We expect that \(C^{-1}MC = D\) is a diagonal matrix. Let’s check this. First, we note that the columns of \(C\) are orthonormal. This implies that \(C\) is an orthogonal matrix so that \(C^{-1} = C^T\). In particular, for \(C\) given above we have \(C^T = C\), in which case \(C^{-1} = C\). Hence,
\[
C^{-1}MC = \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & 2 \\
2 & -1
\end{pmatrix}
\begin{pmatrix}
6 & -2 \\
-2 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
2 & -1
\end{pmatrix}
= \frac{1}{5}
\begin{pmatrix}
1 & 2 \\
2 & -1
\end{pmatrix}
\begin{pmatrix}
2 & 14 \\
4 & -7
\end{pmatrix}
= \begin{pmatrix}
2 \\
0
\end{pmatrix}.
\]

Note that the order of the eigenvectors appearing as columns in \(C\) determines the order of the eigenvalues appearing along the diagonal of
\[
D = \begin{pmatrix}
2 & 0 \\
0 & 7
\end{pmatrix}.
\]

The deformation described by \(D\) is a stretching of the vectors by a factor of 2 along the \(x'\) axis and by a factor of 7 along the \(y'\) axis [cf. eq. (11.19) on p. 152 of Boas].

9. Boas, problem 3.11–42. Verify that the matrix,
\[
H = \begin{pmatrix}
3 & 1 - i \\
1 + i & 2
\end{pmatrix},
\]

is Hermitian. Find its eigenvalues and eigenvectors, write a unitary matrix \(U\) that diagonalizes \(H\) by a similarity transformation, and show that \(U^{-1}HU\) is the diagonal matrix of eigenvalues.

A Hermitian matrix satisfies \(H^\dagger = H\), where \(H^\dagger \equiv H^{* \top}\) is the complex-conjugate transpose of \(H\). Since \((1 - i)^* = 1 + i\), it follows that \(H\) is Hermitian. We compute the eigenvalues by solving the characteristic equation,
\[
\begin{vmatrix}
3 - \lambda & 1 - i \\
1 + i & 2 - \lambda
\end{vmatrix} = (3 - \lambda)(2 - \lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) = 0,
\]

which yields two roots: \(\lambda = 4\) and \(\lambda = 1\). Next, we work out the eigenvectors. Since \(M\) is an Hermitian matrix, we know that the eigenvectors are orthogonal.\(^5\) After normalizing them to

\(^5\)For two complex vectors \(v\) and \(w\), the inner product is defined by \(\langle v, w \rangle = \sum v_i^* w_i\), where \(v_i\) and \(w_i\) are the components of the vectors \(v\) and \(w\). Note the appearance of the complex conjugate, \(v_i^*\), in the expression for the inner product. Two complex vectors are then orthogonal if \(\langle v, w \rangle = 0\). See p. 146 of Boas for further details.
unity, the two eigenvectors are orthonormal. First, we examine:

\[
\begin{pmatrix}
3 & 1 - i \\
1 + i & 2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 4
\begin{pmatrix}
x \\
y
\end{pmatrix},
\]

which yields

\[-x + (1 - i)y = 0, \quad (1 + i)x - 2y = 0.\]

These two equations are not independent, since if you multiply the first equation by \(-1 - i\), you obtain the second equation. It follows that \(x = (1 - i)y\). Normalizing the eigenvector to unity yields:

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}_{\lambda=4}
= \frac{1}{\sqrt{3}}
\begin{pmatrix}
1 - i \\
1
\end{pmatrix}.
\]

The second normalized eigenvector is orthogonal to the first one (keeping in mind the footnotes at the bottom of this page), and thus is given by

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}_{\lambda=1}
= \frac{1}{\sqrt{3}}
\begin{pmatrix}
-1 \\
1 + i
\end{pmatrix}.
\]

One can check this by verifying that,

\[
\begin{pmatrix}
3 & 1 - i \\
1 + i & 2
\end{pmatrix}
\begin{pmatrix}
-1 \\
1 + i
\end{pmatrix}
= \begin{pmatrix}
-1 \\
1 + i
\end{pmatrix}.
\]

The columns of the unitary diagonalizing matrix are given by the orthonormal eigenvectors. Hence,

\[
U = \frac{1}{\sqrt{3}}
\begin{pmatrix}
1 - i & -1 \\
1 & 1 + i
\end{pmatrix}.
\]

Finally, we check that \(U^{-1}HU\) is diagonal. Since \(U\) is unitary, \(U^{-1} = U^\dagger\). Hence

\[
U^{-1}HU
= \frac{1}{3}
\begin{pmatrix}
1 + i & 1 \\
-1 & 1 - i
\end{pmatrix}
\begin{pmatrix}
3 & 1 - i \\
1 - i & 2
\end{pmatrix}
\begin{pmatrix}
1 - i & -1 \\
1 & 1 + i
\end{pmatrix}
= \frac{1}{3}
\begin{pmatrix}
1 + i & 1 \\
-1 & 1 - i
\end{pmatrix}
\begin{pmatrix}
4(1 - i) & -1 \\
4 & 1 + i
\end{pmatrix}
= \begin{pmatrix}
4 & 0 \\
0 & 1
\end{pmatrix}.
\]

As expected, the diagonal elements are the eigenvalues of \(H\).

10. Boas, problem 3.11–58. Consider the matrix,

\[
M = \begin{pmatrix}
5 & -2 \\
-2 & 2
\end{pmatrix}.
\]

Evaluate \(f(M)\) where \(f(M)\) is a series comprised of powers of \(M\). In particular, compute \(M^4\), \(M^{10}\) and \(e^M\).

\footnote{The length of a complex vector \(v\) is given by \(\|v\| = (\sum v_i^* v_i)^{1/2}\). A complex vector \(v\) normalized to unity satisfies \(\|v\| = 1\).}
In Eq. (11.10) on p. 150 of Boas, the diagonalization of \( M \) is performed. The result is:

\[
C^{-1}MC = D, \quad \text{where} \quad C = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.
\]

Then for any function consisting of sums of powers of \( M \), we have

\[
f(M) = f(CDC^{-1}) = Cf(D)C^{-1} = C \text{ diag } (f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)) C^{-1},
\]

where the \( \lambda_i \) are the eigenvalues of \( M \), and diag is a diagonal matrix, whose diagonal entries are given as the arguments. The second step above follows from the observation that

\[
(CDC^{-1})^n = CDC^{-1}CDC^{-1}CDC^{-1} \cdots CDC^{-1} = CD^nC^{-1},
\]

after noting that \( C^{-1}C = I \). Applying Eq. (16) to the matrix given in Eq. (15), we have

\[
f(M) = C \begin{pmatrix} f(1) & 0 \\ 0 & f(6) \end{pmatrix} C^{-1}
\]

\[
= \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} f(1) & 0 \\ 0 & f(6) \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} f(1) & 2f(1) \\ -2f(6) & f(6) \end{pmatrix}
\]

\[
= \frac{1}{5} \begin{pmatrix} f(1) + 4f(6) & 2f(1) - 2f(6) \\ 2f(1) - 2f(6) & 4f(1) + f(6) \end{pmatrix}.
\]

(17)

We apply Eq. (17) to three cases:

\[
M^4 = \frac{1}{5} \begin{pmatrix} 1^4 + 4 \cdot 6^4 & 2 \cdot 1^4 - 2 \cdot 6^4 \\ 2 \cdot 1^4 - 2 \cdot 6^4 & 4 \cdot 1^4 + 6^4 \end{pmatrix} = \begin{pmatrix} 1037 & -518 \\ -518 & 260 \end{pmatrix},
\]

\[
M^{10} = \frac{1}{5} \begin{pmatrix} 1^{10} + 4 \cdot 6^{10} & 2 \cdot 1^{10} - 2 \cdot 6^{10} \\ 2 \cdot 1^{10} - 2 \cdot 6^{10} & 4 \cdot 1^{10} + 6^{10} \end{pmatrix} = \begin{pmatrix} 48372941 & -24186470 \\ -24186470 & 12093236 \end{pmatrix},
\]

\[
e^M = \frac{1}{5} \begin{pmatrix} e^1 + 4e^6 & 2e^1 - 2e^6 \\ 2e^1 - 2e^6 & 4e^1 + e^6 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 + 4e^5 & 2(1 - e^5) \\ 2(1 - e^5) & 4 + e^5 \end{pmatrix}.
\]

11. Boas, problem 3.11–60. The Cayley-Hamilton theorem states that a matrix satisfies its own characteristic equation. Verify this theorem for the matrix

\[
M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.
\]

The characteristic equation of \( M \) is given in eq. (11.4) of Boas,

\[
\begin{vmatrix} 5 - \lambda & -2 \\ -2 & 2 - \lambda \end{vmatrix} = (5 - \lambda)(2 - \lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda - 6)(\lambda - 1) = 0.
\]
Thus, according to the Cayley-Hamilton theorem,

\[ M^2 - 7M + 6I = 0, \]

where \( I \) is the 2 \( \times \) 2 identity matrix and \( 0 \) is the 2 \( \times \) 2 zero matrix. To verify this, we evaluate:

\[
M^2 - 7M + 6I = \begin{pmatrix} 29 & -14 \\ -14 & 8 \end{pmatrix} - 7 \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} + 6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} 29 - 35 + 6 & -14 + 14 \\ -14 + 14 & 8 - 14 + 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Boas suggests an alternate way of verifying the Cayley-Hamilton theorem. First, Boas diagonalizes \( M \) [cf. eq. (11.10) on p. 150]. The result is:

\[ C^{-1}MC = D, \quad \text{where} \quad C = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}. \]

Then, using \( M = CDC^{-1} \), it follows that

\[ M^2 - 7M + 6I = C(D^2 - 7D + 6I)C^{-1}, \tag{18} \]

Noting that the eigenvalues \( \lambda = 6 \) and \( \lambda = 1 \) satisfy the characteristic equation (which is how the eigenvalues are determined in the first place), it immediately follows that

\[ D^2 - 7D + 6I = 0. \tag{19} \]

You can also verify this explicitly:

\[
D^2 - 7D + 6I = \begin{pmatrix} 1^2 - 7 \cdot 1 + 6 & 0 \\ 0 & 6^2 - 7 \cdot 6 + 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Consequently, Eqs. (18) and (19) yield:

\[ M^2 - 7M + 6I = 0, \]

as required by the Cayley-Hamilton theorem.

**Note:** Using the technique above suggested by Boas, it is straightforward to prove the Cayley-Hamilton theorem for any diagonalizable matrix \( M \).

If \( f(\lambda) \) is the characteristic polynomial, then it follows that \( f(\lambda) = 0 \) for any of the eigenvalues of \( M \), since this is how the eigenvalues are determined. Consider the diagonal matrix \( D \) with the eigenvalues on the diagonal. Then, noting that \( D^n \) has values \( \lambda_i^n \) on the \( i \)-th diagonal element and zero on all off-diagonal elements, we have

\[
f(D) = \begin{pmatrix} f(\lambda_1) & 0 & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & 0 & \cdots & 0 \\ 0 & 0 & f(\lambda_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda_n) \end{pmatrix} = 0. \tag{20}
\]
Next one writes $M = CDC^{-1}$ and notes that $M^k = (CDC^{-1})^k = CDC^{-1}(CDC^{-1}) \cdots (CDC^{-1}) = CD^kC^{-1}$, since the internal factors of $C^{-1}C$ cancel. Hence, because $f$ is a polynomial, we have

$$f(M) = f(CDC^{-1}) = Cf(D)C^{-1} = 0$$

since $f(D) = 0$ as shown above. However, this proof is not applicable to a matrix that is not diagonalizable. Nonetheless, one can show that the theorem is valid even for a non-diagonalizable matrix.