1. Boas, Ch. 3, §6, Qu. 6.

The Pauli spin matrices in quantum mechanics are

\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Show that \( A^2 = B^2 = C^2 = I \) (the unit matrix) Also show that any of these two matrices \textit{anticommute}, that is \( AB = -BA \), etc. Show that the commutator of \( A \) and \( B \) is \( 2iC \) and similarly for the other pairs in cyclic order.

\[ A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + (-i) \cdot i & 0 \cdot (-i) + (-i) \cdot 0 \\ i \cdot 0 + 0 \cdot i & i \cdot (-i) + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot i & 0 \cdot (-i) + 1 \cdot 0 \\ 1 \cdot 0 + 0 \cdot i & 1 \cdot (-i) + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \]

\[ BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -AB, \]

\[ AB - BA = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2iC, \]

\[ BC = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \]

\[ CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -BC, \]

\[ BC - CB = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2iA, \]

\[ CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

\[ AC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -CA, \]

\[ CA - AC = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2iB. \]

2. Boas, Ch. 3, §6, Qu. 7.

Find the matrix product of

\[ \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \]

By evaluating this in two ways, verify the associative law for multiplication, \( A(BC) = (AB)C \), which justifies our writing it as \( ABC \) (i.e. without brackets).
Writing

\[ A = \begin{pmatrix} 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \]

we have

\[ AB = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \end{pmatrix}, \]

so

\[ (AB)C = \begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 6. \]

Similarly

\[ BC = \begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 & -4 \end{pmatrix}, \]

so

\[ A(BC) = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ -4 \end{pmatrix} = 6, \]

\[ = (AB)C. \]

3. Boas, problem 3.6–16. Find the inverse of

\[ A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} \]

by employing eq. (6.13) on p. 119 of Boas and also by using Gauss-Jordan elimination.

Eq. (6.13) pm p. 119 of Boas is

\[ A^{-1} = \frac{C^T}{\det A}, \quad (1) \]

where \( C^T \) is the transpose of the matrix whose elements are the cofactors of \( A \).

First we compute the determinant of \( A \) using the expansion of cofactors based on the elements of the third column

\[ \det A = \begin{vmatrix} -2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{vmatrix} = (1) \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} - (2) \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = (1)(4) - (2)(-2) = 8. \]

Next, we evaluate the matrix of cofactors and take the transpose:

\[ C^T = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & -2 \\ 3 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & -3 \\ 1 & -2 & 4 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 6 & -3 & 5 \\ 4 & 2 & 2 \end{pmatrix}. \]
The inverse of $A$ is given by Eq. (1), so it follows that

$$A^{-1} = \frac{1}{8} \begin{pmatrix}
-2 & 1 & 1 \\
6 & -3 & 5 \\
4 & 2 & 2
\end{pmatrix}.$$  

(2)

You should check this calculation by verifying that $AA^{-1} = I$, where $I$ is the $3 \times 3$ identity matrix.

Next we solve the problem by Gauss-Jordan elimination in which we perform the same row operations on the matrix $A$ and the inverse matrix $I$. This proceeds as follows:

$$\begin{pmatrix}
-2 & 0 & 1 \\
1 & -1 & 2 \\
3 & 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

(3)

The left hand matrix, which started as $A$, has been transformed into the identity matrix, so the right hand matrix, which started as the identity matrix, has been transformed into $A^{-1}$. Thus

$$A^{-1} = \frac{1}{8} \begin{pmatrix}
-2 & 1 & 1 \\
6 & -3 & 5 \\
4 & 2 & 2
\end{pmatrix}.$$  

(3)
in agreement with Eq. (2). The Gauss-Jordan method can be efficiently implemented on a computer and requires many fewer operations than using Eq. (1) except for small matrices. By hand, the Gauss-Jordan method involves a lot of copying from line to line which is not needed in the computer program.

4. Boas, problem 3.6–21. Solve the following set of equations by the method of finding the inverse of the coefficient matrix.

\[
\begin{align*}
    x + 2z &= 8, \\
    2x - y &= -5, \\
    x + y + z &= 4.
\end{align*}
\]

The coefficient matrix is:

\[
M = \begin{pmatrix}
    1 & 0 & 2 \\
    2 & -1 & 0 \\
    1 & 1 & 1
\end{pmatrix}.
\]

One can compute the inverse \( M^{-1} \) by using

\[
M^{-1} = \frac{1}{\det M} C^T.
\]  

(5)

First, we compute \( \det M \) via the expansion of cofactors. Using the first row,

\[
\begin{vmatrix}
    1 & 0 & 2 \\
    2 & -1 & 0 \\
    1 & 1 & 1
\end{vmatrix} = (1)(-1) + 2[(2)(1) - (-1)(1)] = 5.
\]

Next, we compute the transpose of the matrix of cofactors,

\[
C^T = \begin{pmatrix}
    -1 & 0 & 0 \\
    1 & 1 & 2 \\
    2 & 1 & 2 \\
    -1 & 1 & 1 \\
    1 & 1 & 1 \\
    1 & 0 & 1 \\
    2 & 1 & 0 \\
    1 & 2 & -1
\end{pmatrix} = \begin{pmatrix}
    -1 & 2 & 2 \\
    -2 & -1 & 4 \\
    3 & -1 & -1
\end{pmatrix}.
\]

Thus, Eq. (5) yields:

\[
M^{-1} = \frac{1}{5} \begin{pmatrix}
    -1 & 2 & 2 \\
    -2 & -1 & 4 \\
    3 & -1 & -1
\end{pmatrix}.
\]

Finally, the solution to the set of equations above is:

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
    -1 & 2 & 2 \\
    -2 & -1 & 4 \\
    3 & -1 & -1
\end{pmatrix} \begin{pmatrix}
    8 \\
    -5 \\
    4
\end{pmatrix} = \begin{pmatrix}
    -2 \\
    1 \\
    5
\end{pmatrix}.
\]

That is, the solution is \( x = -2, y = 1 \) and \( z = 5 \).
5. Boas, problem 3.6–32. For the Pauli spin matrix, \( B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), find \( e^{i\theta B} \) and show that your result is a rotation matrix. Repeat the calculation for \( e^{-i\theta B} \).

First, we observe that:

\[
B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,
\]

where \( I \) is the 2 \( \times \) 2 identity matrix. Thus,

\[
B^{2n} = I, \quad \text{and} \quad B^{2n+1} = B, \quad \text{for any non-negative integer} \ n.
\]

By definition, the matrix exponential is defined via its power series,

\[
e^{i\theta B} = \sum_{n=0}^{\infty} \frac{(i\theta B)^n}{n!} = I + i\theta B - \frac{\theta^2 B^2}{2!} - i\frac{\theta^3 B^3}{3!} + \frac{\theta^4 B^4}{4!} + i\frac{\theta^5 B^5}{5!} + \cdots
\]

\[
= I \left[ 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \right] + iB \left[ \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \right]
\]

\[
= I \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + iB \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = I \cos \theta + iB \sin \theta
\]

\[
= \cos \theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \tag{6}
\]

which is a rotation matrix [cf. Eq.(6.14) on p. 120 of Boas]. The computation of \( e^{-i\theta B} \) follows precisely the same steps, with \( \theta \) replaced everywhere by \(-\theta\). Hence,

\[
e^{-i\theta B} = I \cos \theta - iB \sin \theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

6. Boas, problem 3.7–27. The following matrix,

\[
R \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},
\]

represents an active transformation of vectors in the \( x-y \) plane (axes fixed, vectors rotated or reflected). Show that the above matrix is orthogonal, find its determinant, and find the rotation angle, or find the line of reflection.

The matrix \( R \) is orthogonal, since

\[
RR^T = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.
\]

In addition, the determinant of \( R \) is

\[
\det R = \left| \begin{array}{cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array} \right| = \frac{1}{2} + \frac{1}{2} = 1,
\]

\[5\]
which means that $R$ is a proper rotation matrix. Comparing this with the general form for the matrix representation of an active rotation in two-dimensions,

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix},
\]

we see that $\sin \theta = -\cos \theta = 1/\sqrt{2}$, which implies that $\theta = 3\pi/4 = 135^\circ$.

7. Boas, problem 3.7–34. For the matrix,

\[
L = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix},
\]

find its determinant to see whether it produces a rotation or a reflection. If a rotation, find the axis and angle of rotation. If a reflection, find the reflecting plane and the rotation (if any) about the normal to this plane.

First we compute the determinant of $L$ by using the expansion of cofactors based on the elements of the third row.

\[
\det L = (1) \det \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -1.
\]

Therefore, $L$ is an improper rotation matrix.

If $L$ is a pure reflection matrix (corresponding to a reflection through a plane), then the following two properties must be satisfied:

(a) First, a vector $\vec{r}$ must exist such that $L\vec{r} = -\vec{r}$, since any vector perpendicular to the reflection plane is reversed by the reflection.

(b) Second, two linearly independent solutions to $L\vec{r} = \vec{r}$, perpendicular to the solution of part (a), must exist, since a vector that lies in the reflection plane is unaffected by the reflection.

First, we determine the solution of

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}.
\]

The solution to these equations are $y = z$ and $x = 0$. Thus, the vector $\vec{r} = (0, 1, 1)$ is perpendicular to the plane of reflection.

Is there a rotation in addition to the reflection? By inspection, two vectors in the plane, i.e. perpendicular to the normal, are $(1, 0, 0)$ and $(0, 1, -1)$. One easily sees that they are unchanged by the transformation, i.e. they each satisfy $L\vec{r} = \vec{r}$. Hence there is no rotation in addition to the reflection, so $L$ is a pure reflection matrix.

What would happen, though, if $L$ were not a pure reflection matrix? The most general transformation which has determinant equal to $-1$ corresponds to a reflection about a plane followed by a rotation about the normal to the plane. In this case, to determine the rotation we would have to study how two mutually perpendicular unit vectors in the plane transform into each other under
If the two vectors are $\vec{r}_1$ and $\vec{r}_2$, we compute $L \vec{r}_1$ and $L \vec{r}_2$. We will find that there is an angle $\theta$, the rotation angle, such that

$$L \vec{r}_1 = \cos \theta \, \vec{r}_1 + \sin \theta \, \vec{r}_2,$$

$$L \vec{r}_2 = -\sin \theta \, \vec{r}_1 + \cos \theta \, \vec{r}_2.$$ 

However, for simplicity, in the course I will not set questions which require this additional step.

A special case of the improper rotation matrix is $L = -I$, for which every vector $\vec{r}$ satisfies $L\vec{r} = -\vec{r}$. This transformation represents an inversion through the origin, which can be thought of as reflection through any plane followed by a rotation of 180° about the normal to this plane.

8. Boas, problem 3.8–2. Determine whether the vectors,

$$(1, -2, 3), (1, 1, 1), (-2, 1, -4), (3, 0, 5),$$

are dependent or independent. If they are dependent, find a linearly independent subset. Write each of the given vectors as a linear combination of the independent vectors.

Treat the vectors of Eq. (7) as the rows of a matrix $M$,

$$M = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 1 \\ -2 & 1 & -4 \\ 3 & 0 & 5 \end{pmatrix}.$$ 

We proceed to reduce the matrix $M$ to row echelon form. First, we perform the elementary row operations $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_2 + 2R_1$, and $R_4 \rightarrow R_4 - 3R_1$, and obtain:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{pmatrix}.$$ 

Next, we take $R_3 \rightarrow R_3 + R_2$, which produces a row of zeros. We then switch rows 3 and 4 to obtain:

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 6 & -4 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Next, we take $R_2 \rightarrow R_3 - 2R_2$, which produces a second row of zeros,

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

There are only two non-zero rows so the rank of $M$ is equal to 2. This means that only two of the four vectors of Eq. (7) are linearly independent. In particular, by defining the row vectors corresponding to the first two rows of Eq. (8) as

$$u = (1, -2, 3), \quad v = (0, 3, -2),$$

8
then it is possible to write each vector of Eq. (7) as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. We have

$$\mathbf{y} = a\mathbf{u} + b\mathbf{v} = (a, -2a + 3b, 3b - 2a). \quad (10)$$

Taking $\mathbf{y}$ to be each of the vectors of Eq. (7) in turn, the values of $a$ and $b$ are easily obtained by inspection since $a$ is just the value of the first component of the vector, and the value of $b$ is then obtained by comparing the second components in Eq. (10). The result is

$$(1, -2, 3) = \mathbf{u}, \quad (1, 1, 1) = \mathbf{u} + \mathbf{v},$$

$$(-2, 1, -4) = -2\mathbf{u} - \mathbf{v}, \quad (3, 0, 5) = 3\mathbf{u} + 2\mathbf{v}.$$ \[ \]

One should then verify that that third components on each side of Eq. (10) agree.

9. Boas, problem 3.8–14. Show that the functions $\{e^{ix}, e^{-ix}\}$ are linearly independent.

To solve this problem, we compute the Wronskian (see p. 133 of Boas).

$$W = \begin{vmatrix} e^{ix} & e^{-ix} \\ ie^{ix} & -ie^{-ix} \end{vmatrix} = -2i \neq 0.$$ \[ \]

Since the Wronskian is nonvanishing, the functions $\{e^{ix}, e^{-ix}\}$ are linearly independent.