1. Boas, Ch. 3 §2, Qu. 8.

Consider the following system of equations

\[
\begin{align*}
-x + y - z &= 4 \\
x - y + 2z &= 3 \\
2x - 2y + 4z &= 6
\end{align*}
\]  

(1)

Write and row reduce the augmented matrix to find out whether this set of equations has exactly one solution, no solutions, or an infinite set of solutions. Check your results either by computer or by hand.

The augmented matrix is

\[
\begin{pmatrix}
-1 & 1 & -1 & 4 \\
1 & -1 & 2 & 3 \\
2 & -2 & 4 & 6
\end{pmatrix}
\]

\rightarrow

\[
\begin{pmatrix}
-1 & 1 & -1 & 4 \\
0 & 0 & 1 & 7 \\
0 & 0 & 2 & 14
\end{pmatrix}
\]

\rightarrow

\[
\begin{pmatrix}
-1 & 1 & -1 & 4 \\
0 & 0 & 1 & 7 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(2)

where, in the first transformation we used \( R(3) \rightarrow R(3) + 2R(1), \) \( R(2) \rightarrow R(2) + R(1), \) and in the second transformation we used \( R(3) \rightarrow R(3) - 2R(2). \) The last line of the last expression being all zeros shows that there are an infinite number of solutions. The other two equations are \( z = 7, \) and \(-x + y - z = 4.\) Using the first of these equations the second can be written \( x = y - 11.\)

To verify this solution (given in the boxes) we have

\[
\begin{align*}
-x + y - z &= -11 + 7 = 4, \\
x - y + 2z &= -11 + 14 = 3, \\
2x - 2y + 4z &= -2 \times 11 + 28 = 6,
\end{align*}
\]

(3)

which checks that the equations are satisfied.

2. Boas, Ch. 3 §2, Qu. 9.

Consider the following system of equations

\[
\begin{align*}
x - y + 2z &= 5 \\
2x + 3y - z &= 4 \\
2x - 2y + 4z &= 6
\end{align*}
\]

(4)

Write and row reduce the augmented matrix to find out whether this set of equations has exactly one solution, no solutions, or an infinite set of solutions. Check your results either by computer or by hand.

The augmented matrix is

\[
\begin{pmatrix}
1 & -1 & 2 & 5 \\
2 & 3 & -1 & 4 \\
2 & -2 & 4 & 6
\end{pmatrix}
\]

\rightarrow

\[
\begin{pmatrix}
1 & -1 & 2 & 5 \\
0 & 5 & -5 & -6 \\
0 & -0 & 0 & -4
\end{pmatrix},
\]

(5)
where did the transformations \( R(3) \to R(3) - 2R(1), R(2) \to R(2) - 2R(1) \). The last line gives 0 = -4 which is impossible, so there are no solutions.

3. Boas, Ch. 3 §2, Qu. 12.

Consider the following system of equations
\[
\begin{align*}
2x + 5y + z &= 2 \\
x + y - z &= 1 \\
x + 5z &= 3
\end{align*}
\]

Write and row reduce the augmented matrix to find out whether this set of equations has exactly one solution, no solutions, or an infinite set of solutions. Check your results either by computer or by hand.

The augmented matrix is
\[
\begin{pmatrix}
2 & 5 & 1 & 2 \\
1 & 1 & -1 & 1 \\
1 & 0 & 5 & 3
\end{pmatrix}
\]

where did the transformations \( R(3) \to R(3) - \frac{1}{2}R(1), R(2) \to R(2) - \frac{1}{2}R(1) \), followed by \( R(3) \to R(3) - \frac{5}{3}R(2) \). Solving the resulting equations
\[
\begin{align*}
2x + 5y + z &= 2 \\
-\frac{3}{2}y + \frac{3}{2}z &= 0 \\
2z &= 2
\end{align*}
\]

by back substitution gives \( x = -2, y = 1, z = 1 \).

4. Boas, Ch. 3 §2, Qu. 13.

Consider the following system of equations
\[
\begin{align*}
4x + 6y - 12z &= 7 \\
5x - 2y + 4z &= -15 \\
3x + 4y - 8z &= 4
\end{align*}
\]

Write and row reduce the augmented matrix to find out whether this set of equations has exactly one solution, no solutions, or an infinite set of solutions. Check your results either by computer or by hand.

The augmented matrix corresponding to Eq. (8) is
\[
A = \begin{pmatrix}
4 & 6 & -12 & | & 7 \\
5 & -2 & 4 & | & -15 \\
3 & 4 & -8 & | & 4
\end{pmatrix}
\]
We now proceed to row-reduce \( A \) to row-echelon form. First, we perform the elementary row operations \( R_2 \rightarrow R_2 - \frac{5}{4}R_1 \) and \( R_3 \rightarrow R_3 - \frac{3}{4}R_1 \) to obtain
\[
\begin{pmatrix}
4 & 6 & -12 & \mid & 7 \\
0 & -\frac{19}{2} & 19 & \mid & -\frac{95}{4} \\
0 & -\frac{1}{2} & 1 & \mid & -\frac{5}{4}
\end{pmatrix}.
\]

Next, we perform the elementary row operation \( R_3 \rightarrow R_3 - 19R_2 \) to obtain
\[
\begin{pmatrix}
4 & 6 & -12 & \mid & 7 \\
0 & -\frac{19}{2} & 19 & \mid & -\frac{95}{4} \\
0 & 0 & 0 & \mid & 0
\end{pmatrix}.
\]

The last equation is \( 0 = 0 \) so there are \boxed{an infinite number of solutions}. The other two equations are \( 4x + 6y - 12z = 7, -2y + 4z = -5 \). These can be written as \boxed{x = -2, y = 2z + \frac{5}{2}}.

We can check this solution by substituting these results back into Eq. (8).
\[
\begin{align*}
4(-2) + 6(2z + \frac{5}{2}) - 12z &= -8 + 15 + 12z - 12z = 7, \\
5(-2) - 2(2z + \frac{5}{2}) + 4z &= -10 - 5 - 4z + 4z = -15, \\
3(-2) + 4(2z + \frac{5}{2}) - 8z &= -6 + 10 + 8z - 8z = 4,
\end{align*}
\]

which completes the check.

5. Boas, Ch. 3 §2, Qu. 14. Consider the following system of equations
\[
\begin{align*}
2x + 3y - z &= -2 \\
x + 2y - z &= 4 \\
4x + 7y - 3z &= 11
\end{align*}
\]

Write and row reduce the augmented matrix to find out whether this set of equations has exactly one solution, no solutions, or an infinite set of solutions. Check your results either by computer or by hand.

The augmented matrix corresponding to Eq. (9) is
\[
A = \begin{pmatrix}
2 & 3 & -1 & \mid & -2 \\
1 & 2 & -1 & \mid & 4 \\
4 & 7 & -3 & \mid & 11
\end{pmatrix}
\]
We now proceed to row-reduce $A$ to row-echelon form. First, we perform the row operations $R_2 \rightarrow R_2 - \frac{1}{2} R_1$ and $R_3 \rightarrow R_3 - 2R_1$ to obtain
\[
\begin{pmatrix}
2 & 3 & -1 & \mid & -2 \\
0 & \frac{1}{2} & -\frac{1}{2} & \mid & 5 \\
0 & 1 & -1 & \mid & 15
\end{pmatrix}.
\]
Next, we perform the row operation $R_3 \rightarrow R_3 - 2R_2$ to obtain
\[
\begin{pmatrix}
2 & 3 & -1 & \mid & -2 \\
0 & \frac{1}{2} & -\frac{1}{2} & \mid & 5 \\
0 & 0 & 0 & \mid & 5
\end{pmatrix}.
\]
The last equation is $0 = 5$, which is impossible so the equations are inconsistent since and has no solutions. One also observes that the rank of the augmented matrix is larger than the rank of the coefficient matrix.

6. Boas, Ch. 3 §2, Qu. 18. Find the rank of
\[
M = \begin{pmatrix}
1 & 0 & 1 & 0 \\
-1 & -2 & -1 & 0 \\
2 & 2 & 5 & 3 \\
2 & 4 & 8 & 6
\end{pmatrix}. \tag{10}
\]
We solve this problem by reducing Eq. (10) to row echelon form. First, we perform the row operations $R_2 \rightarrow R_2 + R_1$ followed by $R_3 \rightarrow R_3 - 2R_2$ and $R_4 \rightarrow R_4 - 2R_1$ to obtain
\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & -2 & 0 & 0 \\
0 & 2 & 3 & 3 \\
0 & 4 & 6 & 6
\end{pmatrix}.
\]
Next we perform the row operations $R_3 \rightarrow R_3 + R_2$ and $R_4 \rightarrow R_4 + 2R_1$ to obtain
\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & 3 & 3 \\
0 & 0 & 6 & 6
\end{pmatrix}.
\]
Finally, we perform the elementary row operation $R_4 \rightarrow R_4 - 2R_3$ to obtain

$$
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & -2 & 0 & 0 \\
0 & 0 & 3 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

The matrix above is now in row echelon form. Since it has one row of zeros and three rows with some non-zero entries, we conclude that $\text{rank } M = 3$.

7. Boas, Ch. 3 §3, Qu. 10.

A determinant of a square matrix is called skew-symmetric if $a_{ij} = -a_{ji}$. Show that a skew-symmetric determinant of odd order is zero.

A skew-symmetric matrix $A$ satisfies the condition:

$$A^T = -A,$$

where $A^T$ is the transpose of $A$ (i.e., rows and columns are interchanged). Taking the determinant of both sides of Eq. (11), it follows that any skew-symmetric matrix must satisfy:

$$\det A^T = \det(-A).$$

To determine the consequences of Eq. (12), we first note that:

$$\det A^T = \det A,$$

for any matrix $A$. Next, we recall that if each element of one row of a matrix is multiplied by a number $k$, then the determinant of the resulting matrix is equal to $k$ times the determinant of the original matrix. (See the useful facts about determinants on p. 91 of Boas.) In particular, if $k = -1$, this means that multiplying a row of a given matrix by $-1$ changes the sign of the determinant. Since the matrix $-A$ is obtained from the matrix $A$ by multiplying all rows of $A$ by $-1$, it follows that:

$$\det(-A) = (-1)^n \det A,$$

where $n$ is the number of rows of $A$. Thus, eqs. (12)–(14) imply that

$$\det A = (-1)^n \det A.$$

If $n$ is an odd number, then $(-1)^n = -1$, in which case,

$$\det A = -\det A.$$

which immediately implies that $\det A = 0$. Thus, we conclude that the determinant of any skew-symmetric matrix of odd order is zero.

Note that if $n$ is even, then Eq. (15) implies that $\det A = \det A$, which is a tautology and provides no useful information. The determinant of a skew-symmetric matrix of even order has some very interesting properties, which we do not have time to investigate here. However, if you are interested, do a google search on the term *pfaffian* to learn more about skew-symmetric determinants of even order.
8. Boas, Ch. 3 §3, Qu. 15. Use Cramer’s rule to solve the following two systems of linear equations:

(a) \[
\begin{align*}
  x - 2y + 13 &= 0 \\
  y - 4x &= 17
\end{align*}
\]  

(b) \[
\begin{align*}
  x - 2y &= 4 \\
  5x + z &= 7 \\
  x + 2y - z &= 3
\end{align*}
\]

Given a set of equations,

\[
C \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},
\]  

where \( C \) is an \( n \times n \) coefficient matrix, Cramer’s rule states that the solution to this set of equations is given by

\[
x_i = \frac{\det C^{(i)}}{\det C}, \quad n = 1, 2, 3 \ldots, n,
\]

where \( C^{(i)} \) is obtained by replacing the \( i \)th column of \( C \) with the right hand side of Eq. (18).

(a) Rewrite Eq. (16) as:

\[
\begin{align*}
  x - 2y &= -13 \\
  -4x + y &= 17
\end{align*}
\]

Applying Cramer’s rule to solve the set of equations given by Eq. (16), we obtain the following solutions for \( x \) and \( y \):

\[
x = \frac{\begin{vmatrix} -13 & -2 \\ 17 & 1 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} -13 & -2 \\ 17 & 1 \\ -4 & 1 \end{vmatrix}} = \frac{(-13)(1) - (-1)(17)}{(1)(1) - (-2)(-4)} = \frac{21}{7} = -3,
\]

and

\[
y = \frac{\begin{vmatrix} 1 & -13 \\ -4 & 17 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -13 \\ -4 & 17 \end{vmatrix}} = \frac{(1)(17) - (-13)(-4)}{(1)(1) - (-2)(-4)} = \frac{-35}{7} = 5.
\]

(b) Applying Cramer’s rule to solve the set of equations

\[
\begin{align*}
  x - 2y &= 4 \\
  5x + z &= 7 \\
  x + 2y - z &= 3
\end{align*}
\]
we obtain the following solutions for $x$, $y$ and $z$:

$$x = \begin{pmatrix} 4 & -2 & 0 \\ 7 & 0 & 1 \\ 3 & 2 & -1 \end{pmatrix} = -1 \begin{pmatrix} 4 & -2 \\ 3 & 2 \\ 1 & 2 \end{pmatrix} + (-1) \begin{pmatrix} 4 & -2 \\ 7 & 0 \\ 5 & 0 \end{pmatrix} = \frac{14 - 14}{-4 - 10} = 2,$$

$$y = \begin{pmatrix} 1 & 4 & 0 \\ 5 & 7 & 1 \\ 1 & 3 & -1 \end{pmatrix} = -1 \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} + (-1) \begin{pmatrix} 1 & 4 \\ 5 & 7 \\ 5 & 0 \end{pmatrix} = \frac{1 + 13}{-4 - 10} = -1,$$

$$z = \begin{pmatrix} 1 & -2 & 4 \\ 5 & 0 & 7 \\ 1 & 2 & 3 \end{pmatrix} = -(-2) \begin{pmatrix} 5 & 7 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} - (2) \begin{pmatrix} 1 & 4 \\ 5 & 7 \\ 5 & 0 \end{pmatrix} = \frac{16 + 26}{-4 - 10} = -3,$$

where we have evaluated the first two $3 \times 3$ determinants in the numerator and the $3 \times 3$ determinant in the denominator by expanding in cofactors using the elements of the third column, and the third $3 \times 3$ determinant in the numerator by expanding in cofactors using the elements of the second column.

9. Boas, Ch. 3, §6, Qu. 1.

For

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix},$$

find $AB, BA, A+B, A-B, A^2, B^2, 5A, 3B$. Show that $(A-B)(A+B) \neq (A+B)(A-B) \neq A^2-B^2$. Show that $\det(AB) = \det(BA) = \det(A)\det(B)$ but that $\det(A+B) \neq \det(A) + \det(B)$. Show that $\det(5A) \neq 5\det(A)$ and find $n$ so that $\det(5A) = 5^n\det(A)$. Find similar results for $\det(3B)$.

$$AB = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 10 \\ 1 & 24 \end{pmatrix},$$

$$BA = \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 8 \\ 11 & 21 \end{pmatrix} \neq AB,$$

$$A+B = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix},$$

$$A-B = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} - \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 1 & 1 \end{pmatrix},$$

$$A^2 = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ 16 & 27 \end{pmatrix}.$$
\[
B^2 = \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 2 & 18 \end{pmatrix},
\]

\[
(A - B)(A + B) = \begin{pmatrix} 5 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 4 & 12 \end{pmatrix},
\]

\[
(A + B)(A - B) = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 24 & 6 \end{pmatrix},
\]

\[
A^2 - B^2 = \begin{pmatrix} 11 & 8 \\ 16 & 27 \end{pmatrix} - \begin{pmatrix} 6 & 4 \\ 2 & 18 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 14 & 9 \end{pmatrix} \neq (A - B)(A + B) \neq (A + B)(A - B),
\]

\[
\det(A) = 3 \cdot 5 - 1 \cdot 2 = 13,
\]

\[
\det(B) = (-2) \cdot 4 - 2 \cdot 1 = -10,
\]

\[
\det(AB) = \begin{vmatrix} -5 & 10 \\ 1 & 24 \end{vmatrix} = (-5) \cdot 24 - 10 \cdot 1 = -130 = \det(A) \det(B),
\]

\[
\det(BA) = \begin{vmatrix} -2 & 8 \\ 11 & 21 \end{vmatrix} = (-2) \cdot 21 - 8 \cdot 11 = -130 = \det(A) \det(B),
\]

\[
\det(A + B) = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 1 \cdot 9 - 3 \cdot 3 = 0 \neq \det(A) + \det(B) = 3,
\]

\[
\det(5A) = 15 \cdot 25 - 5 \cdot 10 = 325 \neq 5 \det(A) = 65,
\]

\[
\det(5A) = 325 = 5^2 \det(A) \text{ so } n = 2,
\]

\[
\det(3B) = -6 \cdot 12 - 6 \cdot 3 = -90 \neq 3 \det(A) = -30,
\]

\[
\det(3B) = -90 = 3^2 \det(B) \text{ so } n = 2,
\]