1. Boas, 2.14–9. Evaluate \((-1)^i\) in \(x + iy\) form.

We write 
\[ (-1)^i = e^{i(\pi+2\pi n)} = e^{-\pi-2\pi n}, \]
which, rather surprisingly, is a set of real numbers. The principal value of \((-1)^i = e^{-\pi}\) corresponding to the choice of \(n = 0\) in the above equation.

2. Boas, 2.15–7. Write \(\arctan(i\sqrt{2})\) in \(x + iy\) form.

Start with
\[ z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i} \left( \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \right) = \frac{1}{i} \left( \frac{e^{2iw} - 1}{e^{2iw} + 1} \right). \]

Solving for \(e^{2iw}\),
\[ iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \implies e^{2iw} = \frac{1 + iz}{1 - iz}. \]

Hence,
\[ w = \arctan z = \frac{1}{2i} \ln \left( \frac{1 + iz}{1 - iz} \right). \]

Setting \(z = i\sqrt{2}\), we obtain:
\[ \arctan(i\sqrt{2}) = \frac{1}{2i} \ln \left( \frac{1 - \sqrt{2}}{1 + \sqrt{2}} \right) = \frac{1}{2i} \ln \left[ -(1 - \sqrt{2})^2 \right]. \]

after multiplying the numerator and denominator above by \(1 - \sqrt{2}\) and noticing that \((1 - \sqrt{2})(1 + \sqrt{2}) = 1 - 2 = -1\). We shall make use of the definition of the multi-valued complex logarithm
\[ \ln z = \text{Ln}|z| + i \text{arg} z = \text{Ln}|z| + i(\text{Arg} z + 2\pi n), \quad n = 0, \pm 1, \pm 2, \ldots, \]
with \(z = -(1 - \sqrt{2})^2\). In particular, we have \(\text{Ln}|z| = 2\ln(\sqrt{2} - 1)\) and \(\text{Arg} z = \pi\). Hence, Eqs. (3) and (2) yield:
\[ \arctan(i\sqrt{2}) = (n + \frac{1}{2}) \pi - i\text{Ln}(\sqrt{2} - 1), \quad n = 0, \pm 1, \pm 2, \ldots. \]

3. Boas, 2.16–11. Prove that
\[ \cos \theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n - 1)\theta = \frac{\sin 2n\theta}{2\sin \theta}, \quad (4) \]
\[ \sin \theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n - 1)\theta = \frac{\sin^2 n\theta}{\sin \theta}. \quad (5) \]

Consider the geometric series:
\[ S = e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \cdots + e^{(2n-1)i\theta}. \quad (6) \]
Using the results of eq. (1.4) on p. 2 of Boas,

\[ a + ar + ar^2 + \cdots + ar^{N-1} = \frac{a(1 - r^N)}{1 - r}, \]

we identify \( a = e^{i\theta}, r = e^{2i\theta} \) and \( n = N \). Hence,

\[ S \equiv e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \cdots + e^{(2n-1)i\theta} = \frac{e^{i\theta}(1 - e^{2in\theta})}{1 - e^{2i\theta}}. \]

This result can be simplified by multiply both the numerator and denominator by \(-e^{-i\theta}\). The end result is:

\[ S = \frac{e^{2in\theta} - 1}{e^{i\theta} - e^{-i\theta}} = \frac{e^{i\theta}(e^{in\theta} - e^{-in\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{e^{i\theta} \sin n\theta}{\sin \theta}, \]

where we have used

\[ \sin \theta \equiv \frac{e^{i\theta} - e^{-i\theta}}{2i} \]

to obtain the final result. In particular, using \( \text{Re}(e^{in\theta}) = \cos n\theta \) and \( \text{Im}(e^{in\theta}) = \sin n\theta \), it follows that:

\[ \text{Re } S = \frac{\cos n\theta \sin n\theta}{\sin \theta} = \frac{\sin 2n\theta}{2 \sin \theta}, \]

\[ \text{Im } S = \frac{\sin^2 n\theta}{\sin \theta}, \]

after employing the trigonometric identity \( \sin 2n\theta = 2 \sin n\theta \cos n\theta \). Equating the real and imaginary parts of Eq. (6), we conclude that

\[ \text{Re } S = \cos \theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}, \]

\[ \text{Im } S = \sin \theta + \sin 3\theta + \sin 5\theta + \cdots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin \theta}, \]

which reproduces the results of Eqs. (4) and (5).

4. Boas Ch. 2, §17, Qu. 22. Show that \( \tanh^{-1} z = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right) \).

We write

\[ z = \tanh w = \frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{e^{2w} - 1}{e^{2w} + 1}, \]

so

\[ z \left( e^{2w} + 1 \right) = e^{2w} - 1, \]

which has solution

\[ e^{2w} = \frac{1 + z}{1 - z}. \]

Taking the log gives

\[ w \equiv \tanh^{-1} z = \frac{1}{2} \ln \left( \frac{1+z}{1-z} \right). \]
5. The series for the principal value of the complex-valued logarithm,

\[ \ln(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \]  

(7)

converges for all \( |z| \leq 1, z \neq 1 \). In particular, consider the conditionally convergent series,

\[ S \equiv \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}, \quad \text{where } 0 < \theta < 2\pi. \]  

(8)

Using Eq. (7) with \( z = e^{i\theta} \), it follows that

\[ S = -\ln(1 - e^{i\theta}). \]  

(9)

The principal value of the complex logarithm is given by:

\[ \ln(1 - e^{i\theta}) = \ln|1 - e^{i\theta}| + i\text{Arg}(1 - e^{i\theta}). \]  

(10)

To evaluate this expression we write the complex number \( z = 1 - e^{i\theta} \) in polar form, i.e.

\[ z = 1 - e^{i\theta} = Re^{i\Theta}, \]

so

\[ S = -[\ln R + i\Theta], \quad (-\pi < \Theta \leq \pi). \]  

(11)

A simple way to do this is:

\[ 1 - e^{i\theta} = Re^{i\Theta} = e^{i\theta/2} \left(e^{-i\theta/2} - e^{i\theta/2}\right) = -2ie^{i\theta/2} \sin(\theta/2) = 2e^{i(\theta-\pi)/2} \sin(\theta/2), \]  

(12)

after using \( \sin(\theta/2) = \frac{1}{2i}(e^{i\theta/2} - e^{-i\theta/2}) \) and \(-i = e^{-i\pi/2}\). Since \( \sin(\theta/2) > 0 \) for \( 0 < \theta < 2\pi \), it follows that

\[ R \equiv |1 - e^{i\theta}| = 2 \sin(\theta/2), \]

and we can then identify the principal value of the argument of \( z \) as

\[ \Theta \equiv \text{Arg}(1 - e^{i\theta}) = \frac{1}{2}(\theta - \pi). \]  

(13)

Note that when \( 0 < \theta < 2\pi \), it follows that \( \Theta \) lies between \(-\pi \) and \( \pi \), which is inside the range of the principal value of the argument function as defined in class. Hence, Eq. (11) simplifies to:

\[ S = -\ln(1 - e^{i\theta}) = -\ln \left(2 \sin \frac{\theta}{2}\right) + \frac{1}{2}i(\pi - \theta), \quad \text{for } 0 < \theta < 2\pi. \]  

(14)

(a) By taking the real part of Eq. (8), evaluate

\[ \sum_{n=1}^{\infty} \frac{\cos n\theta}{n}, \quad \text{where } 0 < \theta < 2\pi, \]
as a function of $\theta$. Check that your answer has the right limit for $\theta = \pi$.

Noting that $\cos n\theta = \text{Re} e^{in\theta}$, it follows from Eqs. (8) and (14) that

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = \text{Re} S = -\ln \left(\frac{2 \sin \frac{\theta}{2}}{\theta}\right), \quad \text{for } 0 < \theta < 2\pi.$$  \hspace{1cm} (15)

We can check Eq. (15) in the special case of $\theta = \pi$. Using the results, $\sin(\pi/2) = 1$, $\cos n\pi = (-1)^n$ and $-(-1)^n = (-1)^{n+1}$, it follows that:

\[
\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,
\]
a well-known result.

(b) By taking the imaginary part of Eq. (8), prove that

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{1}{2}(\pi - \theta), \quad \text{where } 0 < \theta < 2\pi.$$  \hspace{1cm} (16)

Noting that $\sin n\theta = \text{Im} e^{in\theta}$, it follows from Eqs. (8) and (14) that

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \text{Im} S = \frac{1}{2}(\pi - \theta), \quad \text{for } 0 < \theta < 2\pi.$$  \hspace{1cm} (16)

Again, we can check the special case of $\theta = \pi$. Since $\sin n\pi = 0$, Eq. (16) reduces in this limit to the trivial equation $0 = 0$.

6. Evaluate the integral

$$I_n = \int_{0}^{\infty} t^n e^{-kt^2} \, dt,$$

for $k > 0$ and $n > -1$.

Let us define a new variable, $u = kt^2$ or equivalently $t = \sqrt{u/k}$. Then,

$$du = 2ktdt \quad \implies \quad dt = \frac{du}{2k} \left(\frac{k}{u}\right)^{1/2} = \frac{du}{2k^{1/2}u^{1/2}}.$$  \hspace{1cm} \hspace{1cm}

Hence,

$$I_n = \frac{1}{k^{n/2}} \int_{0}^{\infty} u^{n/2} e^{-u} \frac{du}{2k^{1/2}u^{1/2}} = \frac{1}{2k^{(n+1)/2}} \int_{0}^{\infty} u^{(n-1)/2} e^{-u} \, du$$

4
Using the definition of the Gamma function,

\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt, \]

we identify \( x = \frac{1}{2}(n + 1) \). Thus,

\[ I_n = \frac{1}{2k^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right) \]

As a check, let us set \( k = 1 \) and \( n = 0 \). Then, we obtain \( I_0 = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi} \) as expected. The factor of \( k^{- (n+1)/2} \) can be deduced using simple dimensional analysis. If \( t \) has dimensions of time, then \( k \) must have dimensions of \( t^{-2} \), since the argument of the exponential must be dimensionless. Since \( dt \) also has dimensions of time, the entire integral has dimensions of \( t^{n+1} \), which must be respected by the final result for \( I_n \). Indeed \( k^{- (n+1)/2} \) has dimensions of \( t^{n+1} \), and so the dimensions of our final result for \( I_n \) are consistent. In particular, it is possible to first carry out the computation with \( k = 1 \), and then determine the \( k \) dependence of \( I_n \) strictly on dimensional grounds!

7. Boas, Ch. 11 §3, Qu. 5. Simplify \( \Gamma\left(\frac{1}{2}\right)\Gamma(4)/\Gamma\left(\frac{9}{2}\right) \).

Using \( x \Gamma(x) = \Gamma(x + 1) \) repeatedly, one obtains \( \Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \Gamma\left(\frac{7}{2}\right) = \frac{7}{2} \cdot \frac{5}{2} \Gamma\left(\frac{5}{2}\right) \), etc. until finally obtaining \( \Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \). Hence, using \( \Gamma(4) = 3! = 6 \), it follows that

\[ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(4)}{\Gamma\left(\frac{9}{2}\right)} = \frac{2}{7} \cdot \frac{2}{5} \cdot \frac{2}{3} \cdot 2 \cdot 6 = \frac{32}{35}. \]

8. Boas, Ch. 11 §3, Qu. 13. Express as a \( \Gamma \) function

\[ \int_0^1 x^2 \left( \ln \frac{1}{x} \right)^3 \, dx. \]

Introduce a new variable \( x = e^{-u} \). Then \( dx = -e^{-u}du \) and \( \ln(1/x) = \ln e^u = u \). Noting that \( x = 0 \implies u = \infty \) and \( x = 1 \implies u = 0 \), it follows that

\[ \int_0^1 x^2 \left( \ln \frac{1}{x} \right)^3 \, dx = \int_0^\infty u^3 e^{-3u} \, du = \frac{1}{3^3} \int_0^\infty v^3 e^{-v} \, dv = \frac{\Gamma(4)}{81} = \frac{2}{27}. \]

In the second step above, I used the overall minus sign to interchange the lower and upper limits of integration. At the third step, I changed the integration variable once more to \( v = 3u \) (the limits of integration are unchanged). Finally, I used the definition of the Gamma function [eq. (3.1) on p. 538 of Boas] followed by \( \Gamma(4) = 3! = 6 \) to obtain the final result.

*The simplest way to see that the argument of any exponential must be dimensionless is to consider the power series expansion, \( e^z = 1 + z + z^2/2! + z^3/3! + \cdots \). Suppose \( z \) had dimensions of length, for example. Then \( e^z \) would be the sum of a dimensionless number plus a number with units of length plus a number with units of squared-length, and so on. But, one cannot consistently sum two quantities that possess different length dimensions. The only way to avoid this inconsistency is to conclude that \( z \) is dimensionless.
9. Boas, problem 11.5-3. Show that the binomial coefficient \( \binom{p}{n} \) can be written in terms of Gamma functions as:

\[
\binom{p}{n} = \frac{\Gamma(p+1)}{n!\Gamma(p-n+1)}.
\]  

(17)

The binomial coefficient is defined in eq. (13.6) on p. 28 of Boas as:

\[
\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}.
\]

To prove that it can be written as in Eq. (17), we first multiply numerator and denominator by \( \Gamma(p-n+1) \),

\[
\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)\Gamma(p-n+1)}{n!\Gamma(p-n+1)}.
\]  

(18)

By repeatedly employing \( x\Gamma(x) = \Gamma(x+1) \), the numerator of Eq. (18) can be written as:

\[
p(p-1)(p-2)\cdots(p-n+3)(p-n+2)(p-n+1)\Gamma(p-n+1) \\
= p(p-1)(p-2)\cdots(p-n+3)(p-n+2)\Gamma(p-n+2) \\
= p(p-1)(p-2)\cdots(p-n+3)\Gamma(p-n+3) \\
= \cdots \\
= p(p-1)(p-2)\Gamma(p-2) = p(p-1)\Gamma(p-1) = p\Gamma(p) = \Gamma(p+1).
\]

Inserting this last result back into Eq. (18) yields

\[
\binom{p}{n} = \frac{\Gamma(p+1)}{n!\Gamma(p-n+1)}
\]

as requested.

10. Boas, Ch. 11 §5, Qu. 5, part (a).

Use eq. (5.4) on p. 541 of Boas to show that

\[
\Gamma\left(\frac{1}{2} - n\right) \Gamma\left(\frac{1}{2} + n\right) = (-1)^n \pi,
\]

(19)

assuming that \( n \) is an integer.\(^\dagger\)

We start from the reflection formula [eq. (5.4) on p. 541 of Boas]:

\[
\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}.
\]  

(20)

\(^\dagger\)Boas restricts \( n \) to be a positive integer. However, Eq. (19) holds for negative integers as well. In fact, noting that \((-1)^n = (-1)^{-n}\) for any integer \( n \), it follows that Eq. (19) is unmodified under the interchange of \( n \) and \(-n\). Note that for \( n = 0 \), Eq. (19) also yields a correct result, \( [\Gamma(\frac{1}{2})]^2 = \pi \).
If we set \( p = \frac{1}{2} - n \), then
\[
1 - p = 1 - \left( \frac{1}{2} - n \right) = n + \frac{1}{2}.
\]

Inserting \( p = \frac{1}{2} - n \) into Eq. (20) yields:
\[
\Gamma \left( \frac{1}{2} - n \right) \Gamma \left( \frac{1}{2} + n \right) = \frac{\pi}{\sin \left[ \pi \left( \frac{1}{2} - n \right) \right]}.
\tag{21}
\]

Finally, we make use of the trigonometric identity,
\[
\sin \left[ \pi \left( \frac{1}{2} - n \right) \right] = \sin \left( \frac{1}{2} \pi - n \pi \right) = \sin \left( \frac{1}{2} \pi \right) \cos(n \pi) - \cos \left( \frac{1}{2} \pi \right) \sin(n \pi) = \cos(n \pi),
\]
after using \( \sin(\frac{1}{2} \pi) = 1 \) and \( \cos(\frac{1}{2} \pi) = 0 \). Using the fact that \( \cos(n \pi) = (-1)^n \) for any integer \( n \), it follows that
\[
\sin \left[ \pi \left( \frac{1}{2} - n \right) \right] = (-1)^n, \quad \text{for any integer } n.
\]

Inserting this result back into Eq. (21) and using \( (-1)^{-n} = (-1)^n \) for any integer \( n \), we obtain:
\[
\Gamma \left( \frac{1}{2} - n \right) \Gamma \left( \frac{1}{2} + n \right) = (-1)^n \pi, \quad \text{for any integer } n,
\]
which is the desired result.