1. [12 points]
Find the interval of convergence of the series
\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x - 1)^n}{n^3}. \]
You must include tests for the end points.

**Note:** Assume that \( x \) is real.

**Ans.**
Use the ratio test.
\[
\rho_n = \left| \frac{(x - 1)^{n+1}}{(n + 1)^3} \cdot \frac{n^3}{(x - 1)^n} \right| = |x - 1| \left( \frac{n}{n+1} \right)^3.
\]
Since the ratio \( n/(n + 1) \) tends to unity for \( n \to \infty \) we have
\[
\rho = \lim_{n \to \infty} \rho_n = |x - 1|.
\]
Hence, according to the ratio test the series converges for \( |x - 1| < 1 \), i.e. for \( 0 < x < 2 \) and diverges for \( |x - 1| > 1 \), i.e. for \( x < 0 \) and \( x > 2 \).

For the end points \( x = 0 \) and \( 2 \) the ratio test is indeterminate. For \( x = 2 \) the series is
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}, \]
which converges by the alternating series test. For \( x = 0 \) the series is
\[ - \sum_{n=1}^{\infty} \frac{1}{n^3}, \]
which converges by the integral test.

Hence the interval of convergence is \( 0 \leq x \leq 2 \).

2. [18 points]

(a) Evaluate
\[ \left( \frac{1 + i}{1 - i} \right)^{100}. \]

**Ans.**
\[
\left( \frac{1 + i}{1 - i} \right)^{100} = \left( \frac{\sqrt{2} e^{i\pi/4}}{\sqrt{2} e^{-i\pi/4}} \right)^{100} = \left( e^{i\pi/2} \right)^{100} = i^{100} = 1^{25} = 1.
\]
(b) Find all values of 
\[(1 + i)^{1/4},\]
expressing your answer in polar form. Your answers should give the principal value of \(\arg z\), i.e. \(\arg z\) should be in the range \(-\pi < \arg z \leq \pi\).

Ans.
\[(1 + i)^{1/4} = \left(\sqrt{2} e^{i(\pi/4+2\pi k)/4}\right)^{1/4} = 2^{1/8} e^{i(\pi/16+\pi k/2)},\]
where \(k\) is an integer. The four distinct solutions which have \(\arg z\) in the range between \(-\pi\) and \(\pi\) have \(k = -2, -1, 0\) and 1, i.e.
\[
\begin{align*}
z &= 2^{1/8} \exp(\pi i/16), \\
z &= 2^{1/8} \exp(9\pi i/16), \\
z &= 2^{1/8} \exp(-7\pi i/16), \\
z &= 2^{1/8} \exp(-15\pi i/16),
\end{align*}
\]

(c) Show that
\[\left| 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{(n-1)i\theta} \right| = \frac{\sin(n\theta/2)}{\sin(\theta/2)}.\]
Verify this result by considering the special case \(\theta = 0\).

Ans.
Summing the geometric series we have
\[
1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{(n-1)i\theta} = \frac{1 - e^{in\theta}}{1 - e^{i\theta}}
= \frac{e^{in\theta/2}(e^{i\theta/2} - e^{-i\theta/2})}{e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})}
= e^{i(n-1)\theta/2} \left[ -2i \sin(n\theta/2) \right]
\]
Taking the absolute value gives
\[
\left| 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{(n-1)i\theta} \right| = \frac{\sin(n\theta/2)}{\sin(\theta/2)}.
\]
For the special case \(\theta = 0\) the sum is equal to \(n\) and the RHS is obtained by taking the limit \(\theta \to 0\). Since \(\sin x = x + \cdots\) for small \(x\) the RHS is \((n\theta/2)/(\theta/2) = n\) in agreement.

3. [12 points]
(a) Express
\[ I_0 = \int_0^\infty e^{-x^3} \, dx \]
and
\[ I_3 = \int_0^\infty x^3 e^{-x^3} \, dx \]
in terms of Gamma functions.

**Ans.**
Let \( y = x^3 \), so \( dy = 3x^2 \, dx \) and hence \( dx = dy/(3y^{2/3}) \). This gives
\[ I_0 = \frac{1}{3} \int_0^\infty \frac{e^{-y}}{y^{2/3}} \, dy = \frac{1}{3} \Gamma \left( \frac{1}{3} \right), \]
\[ I_3 = \frac{1}{3} \int_0^\infty y^{1/3} e^{-y} \, dy = \frac{1}{3} \Gamma \left( \frac{4}{3} \right). \]

(b) Simplify the resulting expression for \( I_3/I_0 \).

**Ans.**
\[ \frac{I_3}{I_0} = \frac{\Gamma(4/3)}{\Gamma(1/3)} = \frac{(1/3)\Gamma(1/3)}{\Gamma(1/3)} = \frac{1}{3}. \]

4. [14 points]
Solve the following set of equations using row reduction (Gaussian elimination):
\[ \begin{align*}
  x + y - z &= 2, \\
  2x - y + 3z &= 5, \\
  3x + 2y - 2z &= 5.
\end{align*} \]

Check your answer by substituting the values for \( x, y \) and \( z \) back into the equations.

**Note:** To get credit you MUST use the method of row reduction.

**Ans.**
The augmented matrix is
\[ \left( \begin{array}{ccc|c}
  1 & 1 & -1 & 2 \\
  2 & -1 & 3 & 5 \\
  3 & 2 & -2 & 5 \\
\end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
  1 & 1 & -1 & 2 \\
  0 & -3 & 5 & 1 \\
  0 & -1 & 1 & -1 \\
\end{array} \right) \rightarrow \left( \begin{array}{ccc|c}
  1 & 1 & -1 & 2 \\
  0 & -3 & 5 & 1 \\
  0 & 0 & -\frac{2}{3} & -\frac{4}{3} \\
\end{array} \right). \]

where, in the first transformation we used \( R^{(3)} \rightarrow R^{(3)} - 3R^{(1)} \), \( R^{(2)} \rightarrow R^{(2)} - 2R^{(1)} \), and in the second transformation we used \( R^{(3)} \rightarrow R^{(3)} - \frac{1}{3}R^{(2)} \). The augmented matrix is now in row echelon form. The corresponding equations are
\[ \begin{align*}
  x + y - z &= 2 \\
  -3y + 5z &= 1 \\
  -\frac{2}{3}z &= -\frac{4}{3}.
\end{align*} \]
Solving by back substitution, starting with the last equation gives $z = 2$, $-3y + 5 \cdot 2 = 1$ so $y = 3$, and $x + 3 - 2 = 2$ so $x = 1$. Hence the solution is $x = 1, y = 3, z = 2$.

Substituting into the original equations gives

\begin{align*}
x + y - z &= 1 + 3 - 2 = 2, \\
2x - y + 3z &= 2 - 3 + 6 = 5, \\
3x + 2y - 2z &= 3 + 2 \cdot 3 - 2 \cdot 2 = 5.
\end{align*}

This verifies that the equations are satisfied.

5. [14 points] Consider the following matrices

\[ A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -7 & 1 \\ 5 & 2 \end{pmatrix} \]

(a) Determine the matrix products $AB$ and $BA$, and hence the commutator $[A, B] \equiv AB - BA$.

\[ AB = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -7 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 26 & -1 \end{pmatrix}, \]

\[ BA = \begin{pmatrix} -7 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -17 & -27 \\ 4 & 22 \end{pmatrix}. \]

\[ [A, B] = \begin{pmatrix} 6 & 10 \\ 26 & -1 \end{pmatrix} - \begin{pmatrix} -17 & -27 \\ 4 & 22 \end{pmatrix} = \begin{pmatrix} 23 & 37 \\ 22 & -23 \end{pmatrix}. \]

(b) The trace, $Tr$, of a matrix is the sum of its diagonal elements. Determine $Tr(AB)$ and $Tr(BA)$.

\[ \text{Ans.} \]

From the answer to the previous part we have $\text{Tr}(AB) = \text{Tr}(BA) = 5$.

(c) Show quite generally that $\text{Tr}(AB) = \text{Tr}(BA)$ for any square $n \times n$ matrices $A$ and $B$.

\[ \text{Note: You must write out the indices and show that you get the same expression for both cases.} \]

\[ \text{Ans.} \]

Now $\text{Tr}(AB) = \sum_i \left( \sum_j A_{ij} B_{ji} \right)$ and $\text{Tr}(BA) = \sum_i \left( \sum_j B_{ij} A_{ji} \right)$. If we interchange the labels $i$ and $j$ in the expression for $\text{Tr}(BA)$ we get the expression for $\text{Tr}(AB)$. Hence, quite generally $\text{Tr}(AB) = \text{Tr}(BA)$.

6. [18 points]

Consider the matrix

\[ H = \begin{pmatrix} 2 & 3 - i \\ 3 + i & -1 \end{pmatrix}. \]
(a) Show that \( H \) is Hermitian.

**Ans.**

A matrix is Hermitian is \( H^\dagger \equiv (H^T)^* = H \), where \( T \) denotes the transpose. Now

\[
H^T = \begin{pmatrix} 2 & 3 + i \\ 3 - i & -1 \end{pmatrix},
\]

and taking the complex conjugate of this gives back \( H \). Hence \( H \) is Hermitian.

(b) Find the eigenvalues and eigenvectors of \( H \).

**Ans.**

The eigenvalues are given by

\[
\begin{vmatrix} 2 - \lambda & 3 - i \\ 3 + i & -1 - \lambda \end{vmatrix} = 0,
\]

which gives \( \lambda^2 - \lambda - 2 - (9 + 1) = 0 \), or \( (\lambda - 4)(\lambda + 3) = 0 \). Hence the eigenvalues are \( \lambda = 4 \) and \( -3 \).

For \( \lambda = 4 \) the eigenvector is given by

\[
\begin{pmatrix} -2 & 3 - i \\ 3 + i & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0,
\]

which has solution \( a = (3 - i)b/2 \). Hence the normalized eigenvector is

\[
\vec{x}^{(\lambda=4)} = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 - i \\ 2 \end{pmatrix}.
\]

For \( \lambda = -3 \) the eigenvector is given by

\[
\begin{pmatrix} 5 & 3 - i \\ 3 + i & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0,
\]

which has solution \( b = (3 + i)a/2 \). Hence the normalized eigenvector is

\[
\vec{x}^{(\lambda=-3)} = \frac{1}{\sqrt{14}} \begin{pmatrix} -2 \\ 3 + i \end{pmatrix}.
\]

(c) Form a matrix \( U \) by stacking the normalized (column) eigenvectors side by side. Show that \( U \) is unitary.

**Ans.**

The matrix \( U \) is

\[
U = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 - i & -2 \\ 2 & 3 + i \end{pmatrix}.
\]

To show that \( U \) is unitary we evaluate \( U U^\dagger \):

\[
UU^\dagger = \frac{1}{14} \begin{pmatrix} 3 - i & -2 \\ 2 & 3 + i \end{pmatrix} \begin{pmatrix} 3 + i & 2 \\ -2 & 3 - i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

i.e. the identity matrix, so \( U \) is unitary.
(d) Verify, by doing the matrix multiplications, that

\[ U^\dagger H U = D \]

where \( D \) is a diagonal matrix with the eigenvalues on the diagonal, and \( U^\dagger \) is the Hermitian conjugate of \( U \).

Ans.

\[
U^\dagger H U = \frac{1}{14} \begin{pmatrix} 3 + i & 2 \\ -2 & 3 - i \end{pmatrix} \begin{pmatrix} 2 & 3 - i \\ 3 + i & 2 \end{pmatrix} \begin{pmatrix} 3 - i & -2 \\ 2 & 3 + i \end{pmatrix} \\
= \frac{1}{14} \begin{pmatrix} 3 + i & 2 \\ -2 & 3 - i \end{pmatrix} \begin{pmatrix} 12 - 4i & 6 \\ 8 & -9 - 3i \end{pmatrix} \\
= \begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix},
\]

i.e. a diagonal matrix with the eigenvalues on the diagonal.

7. [12 points]

(a) State which of the following equations are sensible and explain your reasons (the indices refer to Cartesian components):

\[
\begin{align*}
C_{ijklm} x_{jkm} &= B_{il} \\
F_{ij} x_{ji} &= C \\
M_{ijklm} x_{i} x_{j} x_{k} x_{k} &= C_l \\
\epsilon_{ijk} \frac{\partial G_i}{\partial x_j} &= B_{k} \\
\frac{\partial F_{ijk}}{\partial x_k} &= B_{jm}.
\end{align*}
\]

Note: \( \epsilon_{ijk} \) is the fully antisymmetric third rank (pseudo) tensor discussed in class.

Ans.

In Eq. (1), both sides are the \( il \)-element of a second rank tensor so this is sensible.

In Eq. (2), both sides are scalars so this is sensible.

In Eq. (3), the LHS is the \( lm \)-element of a second rank tensor, and the RHS is the \( l \)-th element of a vector. Hence this equation is not sensible because the two sides have different tensor structure (i.e. the uncontracted indices are different).

In Eq. (4), both sides are the \( k \)-th component of a vector so this is sensible.

In Eq. (5), the LHS is the \( ij \)-element of a second rank tensor, and the RHS is the \( jm \)-th element of a second rank tensor. Hence this equation is not sensible because the two sides have different tensor structure (i.e. the uncontracted indices are different).

(b) In the equation

\[ J_i = \epsilon_{ijk} \omega_j x_k, \]

you are given that \( \vec{x} \) is a polar vector and \( \vec{\omega} \) is a pseudo (axial) vector. Is \( \vec{J} \) a polar or pseudo vector?
\textbf{Ans.}
\vec{\omega} is a pseudo vector and \( \epsilon \) is a pseudo tensor, as stated in part (a). Pseudo tensors have an extra minus sign in the transformation under inversion (an example of an improper rotation) so here we have two extra minus signs which is positive. Hence \( \vec{J} \) is an ordinary (polar) vector, not a pseudo (axial) vector.

(c) Determine whether
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]
describes a proper or an improper rotation.

\textbf{Ans.}
The determinant of the matrix is equal to 
\[
(1)[-(1/\sqrt{2})(1/\sqrt{2}) - (1/\sqrt{2})(1/\sqrt{2})] = -1.
\] Since the determinant is \(-1\), rather than \(+1\), this matrix describes an improper rotation.