Physics 116A
Solving linear equations by Gaussian Elimination (Row Reduction)

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I. INTRODUCTION

The general problem is to solve $m$ linear equations in $n$ variables. In most of this handout we will only consider the important class of problems where the number of equations equals the number of variables, i.e. $n = m$. As we shall see, in this case there is a unique solution unless a particular condition is satisfied. At the end, in Secs. VII and VIII we will discuss the situation when the number of equations is different from the number of variables.

You will have already solved simple examples, for example with two equations in two unknowns, by eliminating one variable, solving for the other, and finally substituting to get the second one. Doing this systematically for arbitrary $n$ turns out to be an efficient way of solving sets of linear equations, and is known as Gaussian elimination. As we shall see, it also corresponds to the row reduction technique described in the handout on determinants, http://young.physics.ucsc.edu/116A/determinants.pdf.

The problem, then, in most of this handout, is to solve the $n$ equations

$$
\sum_{j=1}^{n} a_{ij} x_j = b_i,
$$

where $i = 1, 2, \cdots, n$, for the $n$ unknown quantities $x_i$. The $a_{ij}$ form an $n \times n$ square matrix of coefficients

$$
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix},
$$

and the $b_i$, which form an $n$-component vector ($n \times 1$ matrix), are the specified values on the right hand sides of the equations. The set of equations in Eq. (1) is also sometimes written as

$$
A \vec{x} = \vec{b},
$$

which emphasizes that the $x_i$ and $b_i$ are vectors.
II. EXAMPLE

It will be most convenient to explain the method by solving a particular example which we take to be

\[\begin{align*}
2x_1 + x_2 + 3x_3 &= 4, \\
2x_1 - 2x_2 - x_3 &= 1, \\
-2x_1 + 4x_2 + x_3 &= 1,
\end{align*}\]

which corresponds to

\[A = \begin{pmatrix}
2 & 1 & 3 \\
2 & -2 & -1 \\
-2 & 4 & 1
\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}
4 \\
1 \\
1
\end{pmatrix}.\] (5)

The labels we give to the variables is immaterial and all the information is contained in the elements of the matrix \(A\) and the right hand sides \(\vec{b}\). We therefore focus on the “augmented matrix”, which contains those elements and which we denote by \(A|b\). In the present case it is given by

\[A|b = \begin{pmatrix}
2 & 1 & 3 & 4 \\
2 & -2 & -1 & 1 \\
-2 & 4 & 1 & 1
\end{pmatrix}.\] (6)

One can perform the following operations on the augmented matrix without changing the solution:

i. Interchange rows. This just interchanges the order of the equations.

ii. Add a multiple of another row to a row. This just forms a linear combination of the two equations.

iii. Multiply a row by a constant. This just multiplies the corresponding equation by a constant.

The most important operation will be (ii), with sometimes (i) also being needed. We will not use operation (iii) in the main part of this handout, only in Appendix C.

Note: We will not interchange columns since this would correspond to swapping two of the variables, and we would need to keep track of the swaps to know which variable is which at the end.
III. THE FIRST STAGE: REDUCTION TO ROW ECHELON FORM

The first stage of Gaussian elimination is to use these transformations to put the augmented matrix in “row echelon form”, which means that the entries below the “leading diagonal” are zero. (This definition is slightly simplified, see Appendix A for a more precise definition.) The leading diagonal contains the top element in the first column, the next to top element in the second column, etc. For the present problem the leading diagonal contains elements (2, −2, 1).

To transform the matrix in Eq. (6) to row echelon form we start by putting zeros in the first column below the top element. This requires the following transformations:

\[ R^{(2)} \rightarrow R^{(2)} - R^{(1)}, R^{(3)} \rightarrow R^{(3)} + R^{(1)}, \]

where \( R^{(k)} \) refers to the \( k \)-th row. The augmented matrix then becomes

\[
\begin{pmatrix}
2 & 1 & 3 & 4 \\
2 & -2 & -1 & 1 \\
-2 & 4 & 1 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & 1 & 3 & 4 \\
0 & -3 & -4 & -3 \\
0 & 5 & 4 & 5
\end{pmatrix},
\] (7)

which has zeroes below the top element in the first column as required. Note that the top left element in the augmented matrix in Eq. (6) must be non-zero since it is this element which is being combined with the other elements in the first column in other rows to produce zeros. If the ‘11’ element is zero we first swap the top row with another row which has a non-zero element in the first column. This crucial element (‘11’ in this example) is called a “pivot”. In this handout I indicate a pivot element by putting a box around it and coloring it green. Mathematically the transformations that we have performed to get zeroes below the top element in the first column are

\[ R^{(k)} \rightarrow R^{(k)} - \frac{a_{k1}}{a_{11}} R^{(1)}, \quad (k = 2, 3, \cdots, n). \] (8)

This has to be performed for all the \( n + 1 \) columns so the number of operations is actually \( n - 1 \) (the number of rows that are transformed) times \( n + 1 \) (the number of columns).

Having put zeros below the top element in the first column in Eq. (7) we now transform it to put zeros below the next to top element in column 2. The pivot is the ‘22’-element, and the required transformation is \( R^{(3)} \rightarrow R^{(3)} + \frac{5}{3} R^{(2)} \), which leads to the following augmented matrix

\[
\begin{pmatrix}
2 & 1 & 3 & 4 \\
0 & -3 & -4 & -3 \\
0 & 5 & 4 & 5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
2 & 1 & 3 & 4 \\
0 & -3 & -4 & -3 \\
0 & 0 & -\frac{8}{3} & 0
\end{pmatrix}.
\] (9)
Mathematically the transformation is

$$R^{(k)} \rightarrow R^{(k)} - \frac{a_{kl}}{a_{22}} R^{(2)}, \quad (k = 3, 4, \cdots, n).$$  \hspace{1cm} (10)

Equation (9), in which there are zeroes below the leading diagonal, is in “row echelon form” as stated above. I emphasize that solving the equations represented by the row-echelon matrix in Eq. (9) is equivalent to solving the original set, represented by Eq. (6).

For $n > 3$ we would need additional row transformations to reduce the matrix to row echelon form.

To summarize so far, the first stage of Gaussian elimination is to reduce the augmented matrix to row echelon form by the series of transformations summarized by

$$R^{(k)} \rightarrow R^{(k)} - \frac{a_{kl}}{a_{ll}} R^{(l)}, \quad (k = l + 1, l + 2, \cdots, n),$$  \hspace{1cm} (11)

performed for $l = 1, 2, 3, \cdots, n - 1$ in that order. If, at any stage, the pivot element $a_{ll}$ is zero, then swap the $l$-th row with another row in which the element in the $l$-th column is non zero. The pivot element (the new $a_{ll}$) will then be non-zero.

IV. SECOND STAGE: BACK SUBSTITUTION

The second stage of Gaussian elimination is to solve the row echelon form of the equations. To see that this is now easy we write out the equations corresponding to Eq. (9):

$$\begin{align*}
2x_1 + x_2 + 3x_3 & = 4, \\
-3x_2 - 4x_3 & = -3, \\
-\frac{8}{3}x_3 & = 0.
\end{align*}$$  \hspace{1cm} (12)

I emphasize that these equations are equivalent to the original ones in Eq. (4) but are much easier to solve. The reason is that the last equation in (12) only involves $x_3$ and so can be immediately solved ($x_3 = 0$ here). This can be substituted into the second equation which can then immediately be solved for $x_2$ because $x_2$ is the only unknown (gives $x_2 = 1$ here). Both $x_2$ and $x_3$ are then substituted into the first equation which is immediately solved for $x_1$ since $x_1$ is the only unknown (gives $x_1 = 3/2$ here). Hence the solution to Eq. (4) is

$$x_1 = \frac{3}{2}, \quad x_2 = 1, \quad x_3 = 0.$$  \hspace{1cm} (13)
Solving the equations starting from the last one, and substituting the known values into previous equations, is called back substitution. It forms the second stage of Gaussian elimination. The student should verify that the values in Eq. (13) satisfy the original equations (4).

As an alternative to back substitution, one can continue to row reduce further, putting zeros above the diagonal as well as below it, and also making the diagonal entries equal to one. This process, called Gauss-Jordan elimination, transforms the augmented matrix into what is called “reduced row echelon” form. It is described in Appendix C.

V. OPERATION COUNT

It is useful to estimate, roughly for large \( n \), the number of operations needed to solve \( n \) equations in \( n \) unknowns using Gaussian elimination. In Eq. (11) the number of values of \( k \) and \( l \) is proportional to \( n \) and each row has \( n \) elements that need to be transformed. Hence the number of operations is proportional to \( n^3 \). A more careful analysis shows that the operation count is actually \( n^3/3 \) for large \( n \). In the handout on determinants, http://young.physics.ucsc.edu/116A/determinants.pdf, we showed that the number of operations to evaluate a determinant also varies as \( n^3/3 \) for large \( n \). It is not surprising that we get the same answer because the transformations performed here are the same as the row transformations for determinants except that here we have the extra column on the right in the augmented matrix. However, for large \( n \), the extra work in doing one more column doesn’t change the leading behavior of the operation count.

VI. WHEN IS THERE NOT A UNIQUE SOLUTION (FOR \( m=n \))?

In the example considered here there was a unique solution for the variables \( x_i \). Under what circumstances is this not the case? As an illustration consider the following equations

\[
\begin{align*}
    x_1 - x_2 + 4x_3 &= 5, \\
    2x_1 - 3x_2 + 8x_3 &= 4, \\
    x_1 - 2x_2 + 4x_3 &= 9,
\end{align*}
\]
so

\[
A = \begin{pmatrix}
1 & -1 & 4 \\
2 & -3 & 8 \\
1 & -2 & 4
\end{pmatrix},
\quad
\vec{b} = \begin{pmatrix}
5 \\
4 \\
9
\end{pmatrix}.
\] (15)

The augmented matrix is

\[
A|\vec{b} = \begin{pmatrix}
1 & -1 & 4 & 5 \\
2 & -3 & 8 & 4 \\
1 & -2 & 4 & 9
\end{pmatrix}
\xrightarrow{R(2) \rightarrow R(2) - 2R(1)}
\begin{pmatrix}
1 & -1 & 4 & 5 \\
0 & -1 & 0 & -6 \\
1 & -2 & 4 & 9
\end{pmatrix}
\xrightarrow{R(3) \rightarrow R(3) - R(1)}
\begin{pmatrix}
1 & -1 & 4 & 5 \\
0 & -1 & 0 & -6 \\
0 & 0 & 0 & 10
\end{pmatrix},
\] (16)

where the first transformation is performed by \( R(2) \rightarrow R(2) - 2R(1) \), \( R(3) \rightarrow R(3) - R(1) \), and the second by \( R(3) \rightarrow R(3) - R(2) \). As before, the pivot elements are boxed. The last equation gives

\[
0 = 10,
\] (17)

which is impossible. Hence there are no solutions to Eq. (14).

The reason for their being no solution in the last example is that all the elements of the bottom row of the augmented matrix to the left of the vertical line (i.e. the elements corresponding to the matrix \( A \)) are zero. The action of the row transformations (i) and (ii) in Sec. II that we have carried out on the elements of matrix \( A \), i.e. those elements to the left of the vertical line, leave the determinant of \( A \) unchanged. The determinant of a row reduced square matrix (which, by definition, has zeroes below the diagonal) is the product of the diagonal elements. Hence, if one or more the rows of a row-reduced matrix is all zeros, the determinant is zero. Consequently, Eq. (16) shows that the determinant of matrix \( A \) in Eq. (15) is zero. If you wish you could verify this by a cofactor expansion. Note that if, as part of the allowed row transformations, we had performed operation (iii), i.e. multiplied any of the rows by a constant, this would only have multiplied \( \det(A) \) by that constant so the conclusion that \( \det(A) = 0 \) would have remained unchanged.

We see that there is a unique solution to \( n \) equations in \( n \) variables unless the determinant of the matrix of coefficients \( A \) is zero. We found the same result in class using Cramer’s rule.

If \( \det(A) = 0 \), the last example shows that it is possible to have no solutions. Are there any other possibilities? The answer is yes, and will be illustrated by the next example, which is the
same as the previous one except that it has a different vector of right hand sides $\vec{b}$:

$$\begin{align*}
x_1 - x_2 + 4x_3 &= 1, \\
2x_1 - 3x_2 + 7x_3 &= 4, \\
x_1 - 2x_2 + 3x_3 &= 3.
\end{align*}$$ (18)

The augmented matrix is

\[
\begin{pmatrix}
1 & -1 & 4 & 1 \\
2 & -3 & 7 & 4 \\
1 & -2 & 3 & 3
\end{pmatrix}
\]

\[\rightarrow \begin{pmatrix}
1 & -1 & 4 & 1 \\
0 & -1 & -1 & 2 \\
0 & -1 & -1 & 2
\end{pmatrix}, \quad (19)\]

where, as before, the first transformation is performed by $R^{(2)} \rightarrow R^{(2)} - 2R^{(1)}$, $R^{(3)} \rightarrow R^{(3)} - R^{(1)}$, and the second by $R^{(3)} \rightarrow R^{(3)} - R^{(2)}$. The last equation gives

$$0 = 0,$$ (20)

which is always true. The last equation was previously used to determine $x_3$, whose value was then substituted into the previous equations. Here we see that $x_3$ is undetermined so any value of $x_3$ can be substituted into the first two equations which are $-x_2 - x_3 = 2$, $x_1 - x_2 + 4x_3 = 1$. The solution can be written in terms of the (arbitrary) value of $x_3$ as $x_2 = -2 - x_3$, $x_1 = -1 - 5x_3$. Hence there are an infinite number of solutions to Eq. (18) which depend on one parameter (which we can take to be the value of $x_3$).

The different cases: unique solution, no solution, and an infinite number of solutions, can be usefully understood in terms of the “rank” of the augmented matrix $A|b$ and the coefficient matrix $A$, as discussed in Sec. VIII.

VII. THE SITUATION WHEN THE NUMBER OF EQUATIONS IS DIFFERENT FROM THE NUMBER OF UNKNOWNS

Gaussian elimination can also be applied to the case where the number of equations, $m$, is greater than the number of variables, $n$, (a situation called “overdetermined”) and the case where $m < n$ (called “underdetermined”). We state without proof the following results. For an overdetermined
set of equations, the usual situation is that there are no solutions, but it may occur that there is a unique solution or no solutions at all. For an underdetermined set of equations, the usual situation is that there are an infinite number of solutions, but it may occur that there are no solutions at all. Gaussian elimination deals with all these cases correctly. We now give one example each for the overdetermined and underdetermined cases.

A. Overdetermined equations, \( m > n \)

As an example of overdetermined equations \((m = 3, n = 2)\), we take

\[
\begin{align*}
3x_1 - 2x_2 &= 2, \\
-6x_1 + 3x_2 &= -3, \\
-3x_1 + 5x_2 &= 1.
\end{align*}
\]  

(21)

The augmented matrix is

\[
\begin{pmatrix}
3 & -2 & 2 \\
-6 & 3 & -3 \\
-3 & 5 & 1
\end{pmatrix}
\]

where the first transformation is performed by \( R^{(2)} \rightarrow R^{(2)} + 2R^{(1)} \), \( R^{(3)} \rightarrow R^{(3)} + R^{(1)} \), while the second is performed by \( R^{(3)} \rightarrow R^{(3)} + 3R^{(2)} \). The last equation says \( 0 = 6 \) which is impossible, so there are no solutions. This is the usual situation for overdetermined equations. However, it is also possible that there is a unique solution or an infinite number of solutions, see Sec. VIII.

B. Underdetermined equations, \( m < n \)

As an example of underdetermined equations \((m = 2, n = 3)\), we take

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= 6 \\
-x_1 + 3x_2 + 2x_3 &= -5.
\end{align*}
\]  

(23)
The augmented matrix is

\[
A|b = \begin{pmatrix}
1 & 2 & -1 & 6 \\
-1 & 3 & 2 & -5
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 2 & -1 & 6 \\
0 & 5 & 1 & 1
\end{pmatrix},
\]

where the transformation is performed by \( R^{(2)} \rightarrow R^{(2)} + R^{(1)} \). The second equation is \( 5x_2 + x_3 = 1 \).

One could choose some value for \( x_3 \), so the solution for \( x_2 \) is \( x_2 = \frac{1}{5} - \frac{1}{5} x_3 \), and substitute this into the first equation, \( x_1 + 2x_2 - x_3 = 6 \), which gives \( x_1 = \frac{28}{5} + \frac{2}{5} x_3 \). The conclusion is that there are an infinite number of solutions parametrized by a single parameter which we could take to be \( x_3 \). This is the usual situation with an undetermined set of equations. It is also possible that there are no solutions, see Sec. VIII, though it is not possible to have a unique solution in the underdetermined case.

**C. Comments**

Quite generally, and independent of whether the number of equations is equal to, greater than, or less than, the number of variables, the answer to the question of whether there is a unique solution, no solution, or an infinite number of solutions, can be expressed in terms of the “rank” of the augmented matrix \( A|b \) and the coefficient matrix \( A \), as discussed in Sec. VIII.

**VIII. THE RANK OF A MATRIX**

Which situation actually occurs for a given problem can be determined from the “rank” of the matrix \( A \) and the rank of the augmented matrix \( A|b \). The rank of a matrix, which does not have to be square, is the number of nonzero rows (rows with at least one non-zero element) after the matrix has been reduced to row echelon form. There are other equivalent definitions of rank which are described in Appendix B.

Denoting the rank by \( r \), there is a useful theorem that

\[
r(A) = r(A^T),
\]

where \( A^T \) is the “transpose” of \( A \), i.e. the matrix obtained from \( A \) by interchanging rows and columns. If \( A \) has dimensions \( m \times n \), then \( A^T \) has dimensions \( n \times m \), and, in terms of components, \( A^T_{ij} = A_{ji} \). One consequence of Eq. (25) is that the rank of an \( m \times n \) matrix can not be greater than the smaller of \( m \) and \( n \).
Another useful result follows by noting that the row reduction operations (i) and (ii) in Sec. II, used to get a matrix to row echelon form, leave the determinant unchanged. Operation (iii) simply multiplies the determinant by a constant. The determinant of a matrix with all zeroes below the diagonal is the product of the diagonal elements. If the rank of a square $n \times n$ matrix is less than $n$, then the row reduced matrix has at least one row of zeroes, and hence at least one zero entry on the diagonal. The determinant of an $n \times n$ matrix with rank less than $n$ is therefore zero.

We state without proof the following rules when solving $m$ equations for $n$ unknowns:

- If $r(A) = r(A|b) = n$ there is a unique solution.
- If $r(A) = r(A|b) < n$ there are an infinite number of solutions characterized by $n - r(A)$ parameters.
- If $r(A) < r(A|b)$ there are no solutions.

The motivated student will go through the examples in this handout in Eqs. (9), (16), (19), (22) and (24) to check that they obey these rules. In each of these equations the matrix which has been reduced to row echelon form is boxed.

IX. CONCLUSIONS

To conclude, we have seen that Gaussian elimination, in which one row-reduces the augmented matrix to row echelon form followed by back substitution, is an efficient method for solving linear equations. We considered in detail the important case of $n$ equations in $n$ variables, and saw that there is a unique solution unless the determinant of the $n \times n$ matrix of coefficients of the variables vanishes. In the latter case, there are either no solutions or an infinite number of solutions. We also considered briefly the case when the number of equations is different from the number of equations. In all cases, whether there is a unique solution, or no solutions, or an infinite family of solutions, can be determined from considerations of the rank of the matrix of coefficients $A$ and that of the augmented matrix $A|b$.

Appendix A: The definition of “row echelon” form.

In linear algebra, a matrix is in echelon form if it has the shape resulting of a Gaussian elimination. Specifically a matrix is in row echelon form if:

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes (i.e. all zero rows, if any, belong at the bottom of the matrix).

- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

This is almost the same as having all entries below the principal diagonal equal to zero, but also requires that if there is a zero on the principal diagonal, then all subsequent diagonal entries to the lower right must also be zero. An example of a matrix in row echelon form with some zeroes on the principal diagonal is

$$
\begin{pmatrix}
1 & 2 & 3 & 4 \\
0 & 0 & 7 & 9 \\
0 & 0 & 0 & 6
\end{pmatrix}
$$

(A1)

For most cases that occur in practice, the above rules correspond to simpler statement that a matrix in row echelon form has zeros below the principal diagonal, which is how I expressed it in the text.

For completeness I also give the Wikipedia definition of reduced row echelon form, discussed in Appendix C, which is:

- All nonzero rows are above any rows of all zeroes.

- The leading coefficient of a nonzero row is always strictly to the right of the leading coefficient of the row above it.

- Every leading coefficient is 1 and is the only nonzero entry in its column.

Usually this is equivalent to the statement given in Appendix C, namely that a matrix in reduced row echelon form has elements above and below the principal diagonal which are zero and elements on the principal diagonal are which are unity, but is more general since it allows for the possibility of zeroes on the principal diagonal. For example, the following matrix is in reduced row echelon
Appendix B: Alternative definitions of the “rank” of a matrix

In the text we defined the rank of $A$ to be the number of nonzero rows (rows with at least one non-zero element) after the matrix has been reduced to row echelon form. Two other, equivalent, definitions of rank are also often given. We will state them here without proving that they are equivalent to the one used in the text. Before doing so we note that one can prove that the rank of a matrix is the same as that of its transpose. Consequently, the rank of a matrix $A$ with $m$ rows and $n$ columns can not be greater than the smaller of $m$ and $n$.

- The first of these other definitions involves the notion of a “square submatrix”, which is a square matrix obtained by deleting some of the rows and columns of $A$. If $A$ happens to be square, the whole matrix is one of the submatrices, obtained by deleting no rows or columns. The rank of $A$ can be defined as the size of the largest square submatrix of $A$ which has non-zero determinant.

- In the second of the other definitions of the rank of a matrix, we view the rows of $A$ as $m$, $n$-component vectors. A set of vectors, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ is said to be linearly dependent if there are constants, $c_1, c_2, c_m$, not all zero, such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = 0.$$  \hspace{1cm} (B1)

If the only solution of this equation is all the $c_i$ equal to zero, then the vectors are linearly independent. We will discuss more about linear dependence of vectors a little later in the course. The rank of a matrix can be defined to be the maximum number of linearly independent rows (or, what turns out to be equivalent, the maximum number of linearly independent columns). This definition is not a useful way, in practice, of determining the rank of a matrix. Generally it is used the other way round, namely to determine whether or not a set of vectors are linearly independent by computing the rank of the matrix (formed by the vectors) in some other way, e.g. by reduction of the matrix to row echelon form.
Appendix C: Gauss-Jordan elimination

As an alternative to back substitution, we continue to row reduce further, putting zeros above the diagonal as well as below it, and also making the diagonal entries equal to one. This process is called Gauss-Jordan elimination. We illustrate it by working through the problem in Sec. II.

We start with the augmented matrix in row echelon form shown in Eq. (9). We start by making the element on the bottom right of the coefficient matrix (which here has value \( \frac{8}{3} \)), equal to zero by the row operation \( R_3 \rightarrow -\frac{8}{3}R_3 \).

\[
\begin{pmatrix}
2 & 1 & 3 & 4 \\
0 & -3 & -4 & -3 \\
0 & 0 & -\frac{8}{3} & 0 \\
\end{pmatrix}
\]

We then use the bottom-right element of the coefficient matrix (which now has value 1) as a pivot to create zeros in the elements above it. This involves the row operations \( R_2 \rightarrow R_2 + 4R_3 \), \( R_1 \rightarrow R_1 - 3R_3 \), i.e.

\[
\begin{pmatrix}
2 & 1 & 3 & 4 \\
0 & -3 & -4 & -3 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

We multiply row 2 by \(-\frac{1}{3}\) to make the diagonal element on this row equal to 1 and then use this element as a pivot to make the element above it equal to zero, i.e.

\[
\begin{pmatrix}
2 & 0 & 0 & 3 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

The final step is to multiply row 1 by \( \frac{1}{2} \) to give

\[
\begin{pmatrix}
1 & 0 & 0 & \frac{3}{2} \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

The final form of the augmented matrix is in “reduced row echelon form” in which all elements below and above the principal diagonal are zero, and the elements on the principle diagonal are all unity. (This is the usual situation, but does not apply exactly in all cases. A general definition...
is given in Appendix A.) If the number of variables is equal to the number of variables (as in the example discussed here), the coefficient matrix becomes the identity matrix when transformed into reduced row echelon form, unless its determinant is zero in which case there is one (or more) row of zeros at the bottom. Having the augmented matrix in reduced row echelon form directly solves the problem since the equations corresponding to Eq. (C4) are $x_1 = \frac{3}{2}, x_2 = 1, x_3 = 0$, as found previously by back substitution in Eq. (13). The solution is therefore the right-hand column of the augmented matrix in Eq. (C4).