PHYSICS 242
FINAL EXAM, 2014
Due in my mailbox in ISB 232 by 5 pm. on Tuesday, June 10.

This must be your own work. **No collaboration is allowed.** You may use your notes, books, and the web.

For the C/C++/Fortran/Java questions you must write a program in C, C++, Java or Fortran, but not use Mathematica. (Of course, nothing stops you from checking your answer with Mathematica.) You must provide a listing of the program, the output, and any relevant explanation. Group all parts of a question together.

For the Mathematica questions you must provide a listing of the commands you used, the output, and any relevant explanation. Put all your answers to the Mathematica part in one notebook.

**C/C++/Fortran/Java part**

1. Find the best straight-line fit to the following data:

<table>
<thead>
<tr>
<th>#</th>
<th>x</th>
<th>y</th>
<th>error in y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1075</td>
<td>5.1807</td>
<td>0.0813</td>
</tr>
<tr>
<td>2</td>
<td>2.0198</td>
<td>7.0209</td>
<td>0.0811</td>
</tr>
<tr>
<td>3</td>
<td>2.8948</td>
<td>8.7185</td>
<td>0.0487</td>
</tr>
<tr>
<td>4</td>
<td>3.8009</td>
<td>10.5185</td>
<td>0.0736</td>
</tr>
<tr>
<td>5</td>
<td>4.8840</td>
<td>12.7217</td>
<td>0.0621</td>
</tr>
<tr>
<td>6</td>
<td>6.1507</td>
<td>15.3488</td>
<td>0.1496</td>
</tr>
<tr>
<td>7</td>
<td>7.0411</td>
<td>17.1824</td>
<td>0.0853</td>
</tr>
<tr>
<td>8</td>
<td>8.0592</td>
<td>19.7522</td>
<td>0.1253</td>
</tr>
<tr>
<td>9</td>
<td>8.5211</td>
<td>20.7522</td>
<td>0.1253</td>
</tr>
<tr>
<td>10</td>
<td>10.0766</td>
<td>23.5099</td>
<td>0.1433</td>
</tr>
</tbody>
</table>

(Download this data from [http://young.physics.ucsc.edu/115/homework/qu1fitdata](http://young.physics.ucsc.edu/115/homework/qu1fitdata).)

You need to determine the intercept and slope and the error bars on these quantities.

2. Evaluate

   \[ I = \int_0^2 e^{-x^3} dx \]

numerically to 13 decimal places.

*Note:* You must explain how you estimated that the desired accuracy had been obtained.

3. In my research, I recently needed to find the solution for \( T \) of

   \[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2) \tanh^2 \left( \frac{x}{T} \right) dx = \frac{1}{2}. \]

Solve this equation for \( T \) to 4 decimal places.

*Note:* To integrate to \( \pm\infty \) you may (i) integrate out to some large but finite value and forget about the rest, or (ii) perform a change of variables to make the range of integration finite, or (iii) use the Gaussian quadrature methods that we mentioned briefly in class. To do the root-finding I suggest that you use bisection (safe and sure).

*Note:* Please discuss how you estimated that the desired error had been obtained.

4. Consider the differential equation

   \[ \frac{dy}{dx} = \sqrt{y + x^2}. \]

Find the value of \( y(2) \) correct to 10 decimal places given that \( y(0) = 1 \).

*Note:* You must explain how you estimated that the desired accuracy had been obtained.
5. Compute the following integral by Monte Carlo methods:

\[ I = \int_0^2 dx_1 \int_0^2 dx_2 \cdots \int_0^2 dx_8 \frac{1}{1 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8}. \]

You must indicate how many points you generated and give an estimate for the error bar.

**Mathematica part**

6. (a) Find numerically all the solutions of

\[ x^8 - 4x^3 = 3. \]

(b) Determine how many integers between \(10^6\) and \(1.5 \times 10^6\) are perfect squares, (e.g. like 64 which is equal to \(8^2\).)

(c) Find analytically (using Mathematica) the value of the following integral

\[ \int_0^\infty \frac{1}{(1 + x^n)^m} \, dx, \]

assuming \(n > 0, m > 0, nm > 1\). Find a simpler expression, not involving Gamma functions, for the case of \(m = 1\).

7. In appropriate units, mean field theory predicts that the magnetization \(m\) of an Ising magnet in a magnetic field \(h\) is given by

\[ m = \tanh \left( \frac{Jm + h}{T} \right) \]

You are given that the physical solution is the one with \(m > 0\). Set the interaction \(J\) to be unity.

(a) What is \(m\) at \(T = 1\) for \(h = 0.001\)?

(b) Still with \(h = 0.001\), use a one-line command to plot \(m\) versus \(T\), with \(T\) ranging from a value close to zero up to \(T = 1.3\).

(c) Do a log-log plot of \(m\) against \(h\) at \(T = 1\) (which is the zero-field transition temperature, though you are not required to show that here) for \(h\) between \(10^{-4}\) and \(10^{-1}\).  
   **Note:** There is a Mathematica command `LogLogPlot`.

(d) One can easily show analytically that the correct result for small \(h\) in the previous part is \(m = (3h)^{1/3}\). Verify this by plotting \((3h)^{1/3}\) and \(m\) on the same log-log plot, for the same range as in the previous part.

8. Consider the “sine map”

\[ x_{n+1} = \lambda \sin(\pi x_n), \]

where \(\lambda\) is a parameter in the range \(0 < \lambda \leq 1\), and the \(x_n\) lie in the range from 0 to 1.

(a) Find *analytically* the value of \(\lambda\) at which the fixed point at \(x = 0\) becomes unstable.

(b) Show that the value of \(\lambda\) at which the fixed point at non-zero \(x\) becomes unstable can be expressed as the solution of two coupled equations, for \(\lambda\) and \(x^*\), the fixed point value. Solve these equations numerically.

(c) For each of the following values of \(\lambda\): 0.6, 0.8, 0.85, 0.89, 0.94 and 1, determine the Lyapunov exponent, \(\lambda_L\), and state whether the trajectory is chaotic or not.

(d) For the cases of non-chaotic behavior, determine graphically or otherwise (for example by using the Mathematica function `FixedPoint`) whether one has a fixed point or limit cycle. If you find a limit cycle, determine its length.
(e) For $\lambda = 0.89$ and $x_0 = 1/5$, compute $x_1, x_2$ and $x_{104}$.

9. Consider a quantum mechanical particle in the anharmonic potential

$$\mathcal{H} = \frac{p^2}{2} + \frac{x^2}{2} + x^4.$$ 

(a) Starting from Schrödinger’s equation, use the “shooting method” discussed in class to determine the lowest three energy levels.

(b) Also determine the first ten energy levels using the matrix methods discussed in class, and compare your answer for the first three with the results in part (a).

*Note: You must explain how you tested for convergence.*

*Note: Take $\hbar = m = 1$. 