TWO-DIMENSIONAL SUPERFLUIDS

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ABSTRACT. These lectures describe the behaviour of two-dimensional superfluids, which differ from three-dimensional systems in having phase transitions driven by *thermally excited* vortices. Firstly, I will discuss the Kosterlitz-Thouless-Berezinskii (KTB) theory of the superfluid transition in helium films, which is due to unbinding of vortex-antivortex pairs. Next, I will describe the modifications of this transition due to screening effects at the zero field transition in superconductors. The transition of a clean superconducting film in a perpendicular magnetic field, caused by melting of the flux lattice, will be discussed next. This may also be of the KTB type, involving unbinding of dislocations (with a second transition due to disclination unbinding first discussed by Halperin and Nelson) or may be first order. Finally, the case of a dirty two-dimensional superconductor in a field will be discussed. This has a vortex glass transition (analogous to the spin glass transition in magnetic systems) at zero temperature. Results of recent experiments and simulations which provide evidence for vortex glass behavior will be discussed.

1 Introduction

In bulk superfluids, vortices are lines and have an energy which is proportional to their length (neglecting logarithmic factors coming from long range interactions between different segments of the vortex). The probability of thermally exciting a vortex is therefore very small because its energy is large compared with $k_B T$. Vortices have to be forced into the system, either by applying a field in the case of superconductors [1, 2], or rotating the system in the case of helium [3]. By contrast, in two-dimensional systems, vortices are point defects so the energy to create them is finite (more precisely the energy to create a pair of opposite vortices is finite) and not necessarily very much greater than $k_B T$ so thermal excitation of vortices is quite probable. In fact, as pointed out by Berezinskii [4] and Kosterlitz and Thouless [5] the superfluid transition in two dimensional systems is actually *controlled* by the unbinding of vortex-antivortex pairs.

In these lectures I will discuss the KTB theory as applied to the superfluid transition in ^{4}He, the zero field transition in superconductors, and the transition in a perpendicular magnetic field for clean superconductors. There are several good reviews on the KTB theory, see e.g. [6, 7], and the KTB theory is also discussed at this school by Mooij [8]. My lectures conclude with some recent developments on the transition in a field for dirty superconductors. The disorder destroys both translational [9] and orientational [10] order in the flux lattice, so melting is not expected. Recently, a new type of transition in such systems has been proposed [11] in which the low-T state is a "vortex glass", similar to the spin glass state [12] in magnetic systems, *i.e.* there is no order at any wavevector but the system has a stiffness associated with it so that if it is perturbed in one region, then the state is altered out to infinite distance. Above the transition temperature, T_c, the vortex glass correlation length, the distance over which a local perturbation affects the system, is finite and diverges as T approaches T_c. The necessary conditions for spin glass behavior are randomness and frustration [12] and these indeed occur in disordered type-II superconductors in a magnetic field. In two-dimensions, it appears that the vortex glass transition occurs at $T = 0$ [13, 14], but, as we shall see, the temperature dependence of the correlation length, which diverges as $T \to 0$, leads to observable effects at finite temperature.

Before going into details, it is worth discussing when a film can be considered two-dimensional. If the film thickness is d and the *bulk* correlation length is ξ (for anisotropic systems this would be the correlation length perpendicular to the surface of the film) then the film is effectively two-dimensional when $d < \xi$ and three-dimensional when $d > \xi$. Since the correlation length is temperature dependent and diverges at T_c, a film which is many atoms thick (for a superconductor one would replace the atomic dimension by the Cooper pair size) shows bulk behavior for T significantly above T_c and then crosses over to two-dimensional behavior close to T_c. For really thick films, the 2-d region would be unobservably narrow. On the other hand, films which are only a few atomic layers thick would not show bulk behavior in any region of T since, by the time the bulk correlation length is large enough for the system to be in the critical region, it is already larger than d.

2 The KTB Phase Transition in ^{4}He films

Superfluidity in ^{4}He is believed to be due to Bose-Einstein condensation. The order parameter is therefore the amplitude of the condensate, which has a magnitude and a phase. Long wavelength fluctuations in the phase cost very little energy and so are expected to play an important role. For the moment, let us *only* include such fluctuations. They can be described by the spin-wave Hamiltonian

$$\mathcal{H}_{sw} = \frac{J}{2} \int (\nabla\theta)^2 \, d^2r \quad , \tag{1}$$

where J denotes the interaction strength and θ is the local phase of the condensate. Neglecting the periodicity of θ we can integrate from $-\infty$ to ∞, so the partition function is

$$Z = \int_{-\infty}^{\infty} D[\theta] \exp\left(-\frac{\mathcal{H}}{k_B T}\right) \quad . \tag{2}$$

The spin-wave Hamiltonian is a simplification of a lattice model for the lambda transition in ^{4}He,

$$\mathcal{H}_{XY} = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j) \quad , \tag{3}$$

called the XY model. Unlike the spin-wave Hamiltonian, the XY model includes vortices and spinwave-spinwave interactions and so is much more realistic. In fact, the lambda transition in ^{4}He is believed to be in the same universality class as the XY model.

For the spin-wave Hamiltonian, correlation functions can be evaluated straightforwardly because only Gaussian integrals are involved. One finds

$$\begin{aligned} C(r) &= \langle \exp i[\theta(r) - \theta(0)] \rangle \\ &= \exp\left[-\frac{1}{2} \left\langle (\theta(r) - \theta(0))^2 \right\rangle \right] \quad , \end{aligned} \tag{4}$$

where

$$\begin{aligned} \left\langle (\theta(r) - \theta(0))^2 \right\rangle &= \frac{2k_B T}{J} \int \frac{d^2k}{(2\pi)^2} \left[1 - e^{i\mathbf{k}\cdot\mathbf{r}} \right] \\ &\simeq \frac{k_B T}{\pi J} \ln(r/a_0) \quad , \end{aligned} \tag{5}$$

so

$$C(r) \sim \frac{1}{r^{\eta(T)}} \quad , \tag{6}$$

where a_0 is a short distance cutoff of order the atomic size and

$$\eta(T) = \frac{k_B T}{2\pi J} \quad . \tag{7}$$

In other words, correlation functions decay with a power law, in which the power varies continuously with T. Since power law decay is generally associated with a system at its critical point, this behavior is like a *line* of critical points, called the spin-wave line.

The power law decay in Eq. (7) implies that there is no long range order at finite temperature, a result which can be proved rigorously [15, 16]. On the other hand, a power law decay of the correlations, while reasonable at low temperature, is certainly wrong at high T where the correlations should decay exponentially. One expects, then, that inclusion of additional terms in the Hamiltonian, neglected in the spin-wave approximation, would terminate the spinwave line at a certain temperature T_c, which marks a transition where the decay of the correlations changes from power law to exponential. It turns out that spin-wave interactions do not induce this transition, but rather it is the vortices that do it.

We shall explain the vortex driven transition shortly, but first let us discuss another property of the spin-wave line. Suppose we apply a twist to the phase by suitable boundary conditions, so that there is an *externally imposed* phase gradient $\nabla\Theta$. The extra free energy can be written as

$$\delta F = \frac{S}{2} \int (\nabla\Theta)^2 \, d^2r \tag{8}$$

where S, the stiffness coefficient, is related to the superfluid density, ρ_s, by

$$S = \left(\frac{\hbar}{m}\right)^2 \rho_s \quad , \tag{9}$$

where m is the mass of helium atom. In the spinwave approximation it is easy to see that

$$S = J \quad . \tag{10}$$

Thus, even though there is no long range order, the spin-wave approximation gives a finite superfluid density, so the system is actually a superfluid. Note that it is ρ_s, rather than the order parameter (condensate fraction), ψ, which determines superfluid properties. In mean field theory, these are related by

$$S = J\psi^2 \qquad\qquad \text{MFT} \quad , \tag{11}$$

but in 2-d, fluctuation effects are so big that mean field theory is completely invalid and we have $S > 0$ (below T_c) even though $\psi = 0$ (at all T).

As stated above, T_c is determined by vortices. These are configurations of the local order parameter, $\psi(\mathbf{r}) = |\psi(\mathbf{r})| \exp i\theta(\mathbf{r})$ such that the phase changes by a multiple of 2π if one follows a closed path around a vortex, *i.e.*

$$\oint \nabla\theta \cdot d\mathbf{l} = 2\pi n \quad , \tag{12}$$

where n is an integer. Vortices with $|n| > 1$ have a high energy and can decay into two or more vortices with $|n| = 1$, so we will only consider the cases $n = \pm 1$. At $T = 0$, far away from an isolated vortex the amplitude is constant and we shall set it to be unity by absorbing factors into the definition of J in Eq. (1). However, the amplitude must vanish at the center of the vortex otherwise the vortex energy would be infinite (note that the gradient of the phase is infinite there). The distance over which the amplitude is depressed is called the core size; it is of order the atomic size and we shall denote it by a_0. We shall not need to discuss in detail what happens in the core, which is useful because the core is complicated and not very well understood. According to spin-wave theory, Eq. (1), the energy of a single vortex is

$$E_{\text{1-vortex}} = \pi J \ln(L/a_0) \quad , \tag{13}$$

where L is the linear size of the system. A single vortex therefore costs an infinite energy in the thermodynamic limit. A pair of opposite vortices separated by a distance r has finite energy however:

$$E_{\text{pair}} = 2\pi J \ln(r/a_0) \quad . \tag{14}$$

Because the energy of a pair diverges with r we expect that at low-T all vortices form bound pairs. However, at high-T, the pairs can unbind because of entropy effects. Consider, then, a single vortex. Its entropy is (k_B times) the log of the number of points that the vortex can be put, *i.e.* $k_B \ln(L/a_0)^2$. The *free* energy of a single free vortex is then given by

$$F_{\text{1-vortex}} = (\pi J - 2k_B T)\ln(L/a_0) \quad , \tag{15}$$

which is negative for $k_B T > \pi J/2$. We interpret the temperature at which $F_{\text{1-vortex}}$ changes sign to be T_c, and so, according to this simple 1-vortex argument,

$$k_B T_c = \frac{\pi J}{2} \quad . \tag{16}$$

Despite the simplicity of the argument, we shall see that it contains the essential physics of the problem and gives the transition temperature exactly (including vortex-vortex and spinwave-vortex interactions which have been neglected here) provided the factor of J in Eq. (16) is replaced by the fully renormalized stiffness, S. This is equal to J in the spin-wave approximation, see Eq. (10), but is less than J when screening due to vortices (and also spinwave effects) are included. When the vortices unbind we shall see that $S = 0$ and, as a result, the correlations fall off exponentially with distance.

To describe the KTB transition we need to go beyond the simple case, studied above, of one vortex and treat the many body problem of a finite density of vortices. The vortex Hamiltonian is

$$-\beta\mathcal{H}_V = \frac{2\pi J}{k_B T}\sum_{\langle i,j\rangle} n_i n_j \ln(r_{ij}/a_0) + \ln y \sum_i n_i^2 \tag{17}$$

where, $n_i = \pm 1$,

$$y = e^{-E_0/k_B T} \tag{18}$$

is a fugacity for vortices, E_0 is the core energy, and the total vorticity must be zero, *i.e.*

$$\sum_i n_i = 0 \quad , \tag{19}$$

otherwise the energy diverges. The vortex Hamiltonian, Eq. (17), can be derived microscopically if the cosine in the XY model, Eq. (3), is replaced by the form in the periodic Gaussian or Villain [17, 18] model. (The precise form of the function should be irrelevant). The parameter J in Eq. (17) now includes effects of spin-wave vortex interactions which exist in real ^{4}He films and the XY model, but which vanish for the Villain model. To represent the core, no two vortices can approach closer than the core size a_0.

The vortex Hamiltonian is equivalent to the 2-d Coulomb gas

$$\mathcal{H}_{CG} = -\sum_{\langle i,j\rangle} q_i q_j \ln(r_{ij}) \tag{20}$$

with

$$q_i = (2\pi J)^{1/2} n_i \tag{21}$$

the i-th charge, (note that $q_i^2 \equiv q^2 = 2\pi J$) and so the scalar potential, ϕ, satisfies

$$\nabla^2 \phi = -2\pi\rho \tag{22}$$

with

$$\rho(r) = \sum_i q_i \delta(r - r_i) \quad . \tag{23}$$

In this representation, the system is an insulator at low-T because the charges form neutral "molecules", so the dielectric constant, ϵ, is finite. As we shall see, the stiffness is given by

$$S = \frac{J}{\epsilon} \tag{24}$$

which is finite, so the system is a superfluid. At high temperature there are free charges and so the dielectric constant diverges. Consequently the renormalized stiffness vanishes and the system is in the normal state.

The KTB theory considers the screening effect that *smaller* pairs have on larger pairs. Consider a pair separated by a distance r. The force between them is reduced from $2\pi J/r$ to $2\pi J/(r\epsilon(r))$, where $\epsilon(r)$ is a *scale dependent* dielectric constant representing the screening effect of pairs with separation r' less than r. One determines $\epsilon(r)$ in the usual way from electrostatics,

$$\epsilon(r) = 1 + 2\pi\chi(r) \quad , \tag{25}$$

where the susceptibility, $\chi(r)$, is related to the polarizability of a pair, $\alpha(r)$, and the density of pairs, $n(r)$, by

$$\chi(r) = \int_{a_0}^{r} n(r')\alpha(r')dr' \quad . \tag{26}$$

The polarizability is given by the standard expression

$$\alpha(r) = \frac{(qr)^2}{2k_B T} \tag{27}$$

where the factor of $1/2$ comes from integrating over the direction of the dipole in two-dimensions. The vortex density is given by

$$n(r) = 2\pi r \frac{y^2}{a_0^4} e^{-V(r)/k_B T} \tag{28}$$

where $V(r)$, the energy between two vortices, is obtained by integrating up the force, *i.e.*

$$V(r) = q^2 \int_{\ln a_0}^{\ln r} \frac{d\ln r'}{\epsilon(r')} \quad . \tag{29}$$

These equations enable one to determine $\epsilon(r)$ self-consistently. To simplify the analysis let us define a logarithmic scale, l, by $r = a_0 e^l$ and then define

$$K(l) = \frac{J}{k_B T \epsilon(r = a_0 e^l)} \tag{30}$$

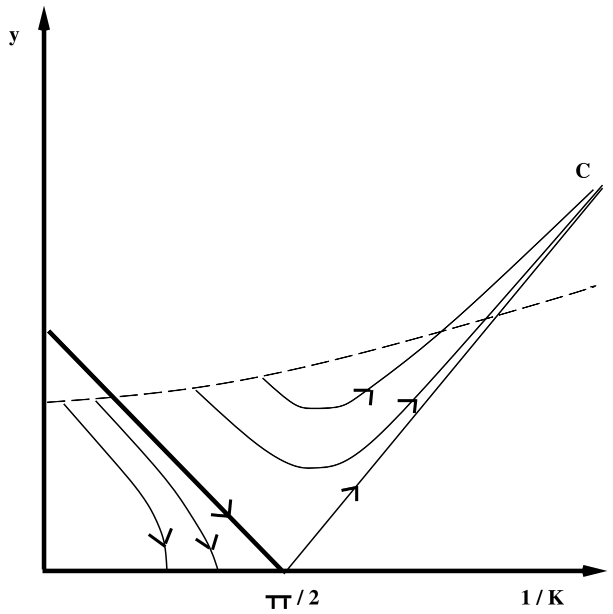

Figure 1: Sketch of solutions of the Kosterlitz equations, Eqs. (32). The dashed line is a locus of physical values of the parameters, on which integration of the equations is started. The solid line terminating at $y = 0, K^{-1} = \pi/2$ is the phase boundary separating the superfluid phase to the left from the normal phase to the right. In the superfluid region the equations flow to the spinwave line, $y = 0$. In the normal region the equations flow to large y.

$$y^2(l) = y^2 \left(\frac{r}{a_0}\right)^4 e^{-V(r)/k_BT} \quad , \tag{31}$$

where $K(l)$ is the effective stiffness (divided by k_BT) on scale l and $y(l)$ is an effective scale dependent fugacity. Equations (25)-(29) can then be reexpressed as

$$\begin{aligned} \frac{dK^{-1}(l)}{dl} &= 4\pi^3 y^2(l) \\ \frac{dy(l)}{dl} &= [2 - \pi K(l)]y(l) \end{aligned} \tag{32}$$

which were first obtained by Kosterlitz [19], though the original Kosterlitz-Thouless theory [5] is equivalent to them, see [20]. Eqs. (32) have to be solved subject to the boundary conditions

$$\begin{aligned} K(l=0) &= \frac{J}{k_BT} \\ y(l=0) &= y \quad . \end{aligned} \tag{33}$$

The right hand side of Eqs. (32) is just the first term in an expansion in powers of $y(l)$, so the density has been assumed to be small.

Eqs. (32) can be integrated analytically for $[2 - \pi K(l)] \ll 1$ and the result is shown in Fig. 1. The thick line, which terminates at $y = 0, K^{-1} = \pi/2$ is the locus of critical points. If the physical values of K and y, *i.e.* $K(l = 0)$ and $y(l = 0)$, lie to the left of this line, then the trajectory flows to the line $y = 0$, which corresponds to there being no vortices on long length scales. This is expected since all the vortices form bound pairs in this region. In other words the long distance physics is described by the spin-wave line but with the effective value of J reduced from its bare value, *i.e.*

$$J \longrightarrow S = \frac{J}{\epsilon} \tag{34}$$

where

$$\epsilon \equiv \epsilon(r = \infty) = \frac{K(l = 0)}{K(l = \infty)} \quad . \tag{35}$$

Thus the spin-wave theory can be taken over but with J replaced by the stiffness S. Since ϵ is finite to the left of the solid line in Fig. 1, the stiffness is finite so this region is a superfluid.

By contrast, if the starting values of y and K^{-1} lie to the right of the thick line in Fig. 1, then the trajectories flow to large values of y, presumably indicating the presence of free vortices, where the equations are inapplicable. The strategy then is to integrate the equations not up to $l = \infty$ but only to a point (C in Fig. 1) where $y(l)$ is still less than unity but the system is far from the critical point. One can then calculate physical quantities from some other theory, such as Debye-Huckel theory, which should be fairly reliable in this region. Eqs. (32) are then used to relate the parameters in the Debye-Huckel theory to the original parameters in the Hamiltonian, see *e.g.* [21].

According to Debye-Huckel theory,

$$\epsilon(\mathbf{k}) = 1 + \frac{2\pi n_f q^2}{k^2} \tag{36}$$

for small $\mathbf{k}$, where $\mathbf{k}$ is the wavevector, q the magnitude of the charge, and n_f the density of free charges. The separation between the free charges must be comparable with the correlation length above T_c, *i.e.*

$$n_f \sim \xi_+^{-2} \quad , \tag{37}$$

and integration of Eqs. (32) gives [19]

$$\xi_+ \sim e^{C/t^{1/2}} \quad , \tag{38}$$

where $t = (T - T_c)/T_c$ and C is a non-universal constant.

From Fig. 1 it is apparent that the critical temperature, T_c, is where

$$\frac{S(T_c^-)}{k_B T_c} = K(l = \infty) = \frac{2}{\pi} \quad , \tag{39}$$

which is to be compared with the simple energy-entropy argument above for a single vortex, Eq. (16), which can be expressed in a similar manner as $K(l=0)=2/\pi$. From Eqs. (39) and (9) it follows that [22]

$$\left(\frac{\hbar}{m}\right)^2 \frac{\rho_s(T_c^-)}{k_B T_c} = \frac{2}{\pi} \quad , \tag{40}$$

so that ρ_s has a universal jump at T_c (remember that $\rho_s = 0$ for $T > T_c$). This prediction has been verified in experiments on helium films [23], though to make a precise comparison one has to generalize the KTB theory to dynamics [24].

Note that the KTB theory is able to make *exact* predictions for the critical behavior because this is determined by the spin-wave line ($y = 0$) where the physics is simple. If the initial value of y is small, then the renormalized fugacity $y(l)$ is always small (at least at and below T_c) so the theory is reliable. Suppose, though, that the initial value of y is not small. Then there are two possibilities. The first is that, below T_c, the fugacity decreases with increasing l down to the region where the theory is applicable and the behavior is still given by the KTB theory. The other possibility is that some other physics may intervene while $y(l)$ is still large. One recent interesting suggestion [25, 26] is that there may be a vortex-antivortex solid in a region of the phase diagram with y large. However, at present, the phase diagram of the Coulomb gas for large fugacity is uncertain.

3 Superconducting Films in Zero Field

Several authors [27, 28, 29] have applied the KTB theory to the superconducting transition in thin films. The main difference compared with helium films is the effect of screening, as a result of which the interaction between vortices only varies as $\ln(r)$ for r less than the screening length, and vanishes at larger r. For bulk superconductors, the screening length is just the London penetration length, λ_L, [1, 2]. However, for films whose thickness, d, is less than λ_L, the effective screening length is [1, 30]

$$\Lambda_{\text{eff}}(T) = \frac{2\lambda_L^2}{d} \quad , \tag{41}$$

which is much larger. This larger screening length arises because the magnetic field, responsible for the screening, is largely *outside* the film. In principle, then, there are always some free vortices so the KTB transition is rounded. However, we shall see that the value of $\Lambda_{\text{eff}}(T_c^-)$ is enormous so the effects of rounding are probably too small to be observed. For practical purposes, therefore, the transition will be sharp.

According to Pearl [30] the interaction between two vortices is

$$U(r) = \frac{\pi}{2}\frac{n_s\hbar^2}{m}\ln(r) + \text{const.} \qquad (r \ll \Lambda) \quad , \tag{42}$$

where m is the electron mass and n_s is the density of superconducting electrons per unit *area*. For short distances one should use the bare value of n_s in this expression but at long distances one should put the fully renormalized value, including screening effects of vortices. According to Eq. (39), the KTB theory predicts that $2\pi S$, the coefficient of $\ln r$ at distances large compared with microscopic lengths but small compared with Λ_{eff}, is $4k_BT_c$. Hence the analogue of the universal jump in Eq. (40) is

$$\frac{n_s(T_c^-)}{k_BT_c}\frac{\hbar^2}{m} = \frac{8}{\pi} \quad . \tag{43}$$

This can be checked experimentally from the low frequency form of the conductivity:

$$\operatorname{Im}\sigma(\omega) = \frac{e^2 n_s}{m\omega} \quad . \tag{44}$$

Eq. (43) does not appear to be satisfied in all systems studied so far [26, 31, 8].

The universal jump in n_s at T_c can also be related to the screening length, Λ_{eff}. Noting that the London penetration depth is related to the *bulk* density of superconducting electrons, n_s/d, by

$$\lambda_L^2 = \frac{mc^2}{4\pi(n_s/d)e^2} \tag{45}$$

we have

$$\Lambda_{\text{eff}}(T) = \frac{mc^2}{2\pi n_s e^2} \tag{46}$$

and so

$$\Lambda_{\text{eff}}(T_c^-) = \Lambda_{T_c} \tag{47}$$

where the "fluctuation length", Λ_T, is given by

$$\Lambda_T = \frac{\phi_0^2}{16\pi^2}\frac{1}{k_BT} \quad , \tag{48}$$

with $\phi_0 = hc/2e$, the flux quantum. Numerically [32]

$$\Lambda_T = \frac{2}{T} \tag{49}$$

where Λ_T is measured in centimeters and T in Kelvin. Thus typically $\Lambda_{\text{eff}}(T_c^-)$ is of order 1 cm, a macroscopic length which might be even larger than the sample size. Thus, any rounding due to the screening length being finite should be unobservably small.

It is important to discuss the size of the critical region, ΔT, *i.e.* the range of temperature over which one can observe the KTB behavior. For clean samples, *i.e.* those where the Cooper pair size, ξ_0, is much smaller than the mean free path l, it turns out to be very small [27, 28]. One way to see this is to assume that the critical region is roughly the difference between T_c and the mean field transition temperature, T_c^{MF}. Using the mean field divergence of λ_L one has roughly

$$\frac{\Lambda_{\text{eff}}(T)}{\Lambda_{\text{eff}}(T=0)} \sim \frac{T_c^{MF}}{T_c^{MF} - T} \tag{50}$$

so

$$\frac{\Delta T}{T_c} \sim \frac{T_c^{MF} - T_c}{T_c} \sim \frac{\Lambda_{\text{eff}}(0)}{\Lambda_{\text{eff}}(T_c{}^-)} \tag{51}$$

which is very small even for films only a few angstroms thick. It is larger, however, for high-T_c systems, because $\Lambda(T_c^-)$ is smaller, see Eqs. (47) and (49). For dirty films, $\Lambda(0)$ is increased and hence so is the critical region. Beasley *et al.* have shown that, in this limit,

$$\frac{\Delta T}{T_c} = 0.17 \frac{e^2}{\hbar} \frac{1}{\sigma_n} \quad , \tag{52}$$

where σ_n is the normal state conductivity. Thus the critical region is significant for films sufficiently dirty that the σ_n is of order the quantum of conductance $e^2/\hbar$.

Next we discuss, following Halperin and Nelson [27], how the dc conductivity diverges as $T \to T_c^+$. The voltage is related to the rate of change of the phase difference, $\Delta\theta$, across the sample by the Josephson relation

$$V = \frac{\hbar}{2e} \frac{d\Delta\theta}{dt} \quad . \tag{53}$$

The change of the phase difference is due to the flow of vortices, *i.e.*

$$\frac{d\Delta\theta}{dt} = 2\pi L n_f v_{\text{vortex}} \tag{54}$$

where $n_f \sim \xi_+^{-2}$ is the density of free vortices, L is the sample size, and the mean vortex velocity, v_{vortex}, is related to the Lorentz force,

$$f_L = \frac{\phi_0}{c} J \quad , \tag{55}$$

by

$$v_{\text{vortex}} = \mu f_L \tag{56}$$

where J is the current density and μ the vortex mobility. Hence we have

$$E = \frac{\hbar^2 \pi^2}{e^2} n_f \mu J \tag{57}$$

and so

$$\sigma = \frac{e^2}{\hbar^2 \pi^2} \frac{1}{n_f \mu} \quad . \tag{58}$$

According to the dynamical theory [24] μ is a constant near T_c and so

$$\sigma \sim \frac{1}{n_f} \sim \xi_+^2 \sim e^{2C/t^{1/2}} \quad . \tag{59}$$

Somewhat further away from T_c one expects [27] that the relation $\sigma \sim \xi_+^2$ will still hold but with ξ_+ replaced by its mean field form, $\xi \sim (T - T_c^{MF})^{-1/2}$, which gives, for the resistance, $R \sim T - T_c^{MF}$. This result is in agreement with the Aslamazov-Larkin theory [33] which calculates the *leading* fluctuation correction as T is reduced towards T_c.

At and below T_c the linear resistance vanishes. In this region, the electric field varies as J to a power greater than unity [27] *i.e.*

$$E \sim J^{x(T)} \tag{60}$$

where $x(T_c^-) = 3$ and x increases as T decreases below T_c. Because of this power law behavior, different from the exponential dependence expected below T_c in 3-d, see *e.g.* [34], the critical current density, *i.e.* the value of J where the dissipation rapidly increases, is zero in 2-d.

We conclude this section by noting that disorder is not expected to change the universal behavior at the KTB transition in superconducting films at zero magnetic field. This is because, according to the Harris [35] criterion, weak disorder is irrelevant if $\nu_{\rm pure} > 2/D$ where D is the space dimension and $\nu_{\rm pure}$ is the correlation length exponent of the pure system. At the KTB transition, the correlation length diverges exponentially, see Eq. (38), which corresponds to $\nu_{\rm pure} = \infty$, so disorder is strongly irrelevant. We shall see, however, that disorder *does* strongly affect the transition in a magnetic field discussed below. This is because disorder acts like a random *field* on the vortex lattice and this is always relevant at the critical point [36]. For $D < 4$, disorder is also relevant at the $T = 0$ fixed point describing the vortex lattice phase and so destroys the vortex lattice [9] in two and three dimensions.

4 Clean Superconducting Films in a Field

In a clean sample of *bulk* type-II material, a flux lattice forms for magnetic field, H, between the critical fields H_{c_1} and H_{c_2} [1, 2]. For the case of two-dimensional thin films, the flux always penetrates and so $H_{c_1} = 0$ [37]. Furthermore, because fluctuations are more important in 2-d systems than in the bulk, there is no sharp transition at the mean field H_{c_2} but this is rather a crossover region where superconducting short range order starts to develop. Just below and to the left of this region, it makes sense to talk about the instantaneous positions of vortices but there is no long range order, so the system can be described as a "vortex liquid", see Fig. 2. At lower temperatures and fields the vortices form a solid triangular array. The transition in a field therefore corresponds to a melting of this "vortex solid" [37, 38].

The properties of this solid phase in two dimensions are very analogous to the properties of the superfluid films discussed above. Firstly, there is no true long-range translational order but density-density correlations decay with a power of the distance. Consequently the 2-d solid is not a crystal. Nonetheless, it is a solid since there is a finite "stiffness", which in this case is just the shear modulus. As noted by Kosterlitz and Thouless [5] melting of the solid might take place by unbinding of dislocation pairs, in a very similar manner to unbinding of vortex-antivortex pairs in the superfluid transition. A major advance in the theory of 2-d melting was made by Halperin and Nelson [39], who noted that solid phase has true long-range *orientational* order and that the phase just above the dislocation unbinding transition is not a liquid, because it has power-law, rather than exponential, decay of orientational correlation functions. They called this new state a "hexatic" because it has 6-fold orientational correlations analogous to the 2-fold orientational correlations in a nematic liquid crystal. According the the KTBHN theory there is then a second KTB type transition from hexatic to liquid involving the unbinding of another type of topological defect, the disclination. In addition to the scenario just presented, where melting proceeds by two KTB transitions, it is also possible that melting proceeds by a single first order transition, direct from the solid to the liquid. Continuous melting has been observed in liquid crystal films, but simulations of atoms in 2-d have generally indicated a first order transition, though with much larger fluctuations than occur in three-dimensional melting. In particular, for the melting of vortices, recent Monte Carlo simulations [40] indicates a first order transition, though earlier work [41, 42] also appeared consistent with the KTBHN scenario, and it has even been claimed that that there is *no* transition [43].

Assuming that the vortex solid melting transition is of the KTBHN type, the phase diagram has been discussed in refs. [37, 38]. Figure 2 shows a sketch based on the results of Fisher [37]. The different phases are the normal state, which crosses over smoothly to the vortex liquid at around the mean field H_{c_2}, the hexatic phase and the vortex solid phase. The hexatic phase would be difficult to detect because its transport properties are essentially the same as those in the vortex liquid region. At the melting line there is a universal jump in a combination of elastic constants, analogous to the universal jump in ρ_s in helium films. To be precise one has

$$\frac{\mu B}{k_B T_m(\mu + B)} a_0^2 = 4\pi \quad , \tag{61}$$

where μ is the shear modulus, B the bulk modulus, T_m the melting temperature, and a_0 is the lattice spacing. For the vortex lattice, B is in general much larger than μ in which case Eq. (61) shows that $\mu(T_m)/T_m$ is universal. Eq. (61) has been used [37] to determine the form of the melting line in Fig. 2. Experimental evidence for melting of the vortex solid close to the point expected from the KTBHN theory has been found [44, 45].

5 The Vortex Glass

As first shown by Larkin [9] disorder destroys translationsl order in the flux lattice for dimension, D, less than four, on scales larger than the "Larkin length", L_P, which depends on the strength of the disorder and tends to infinity in the clean limit. Orientational order is also destroyed by disorder [10], though the orientational correlation length can be much longer than the translational correlation length, see for example [46]. Thus there can be no hexatic or vortex solid phase in the presence of disorder.

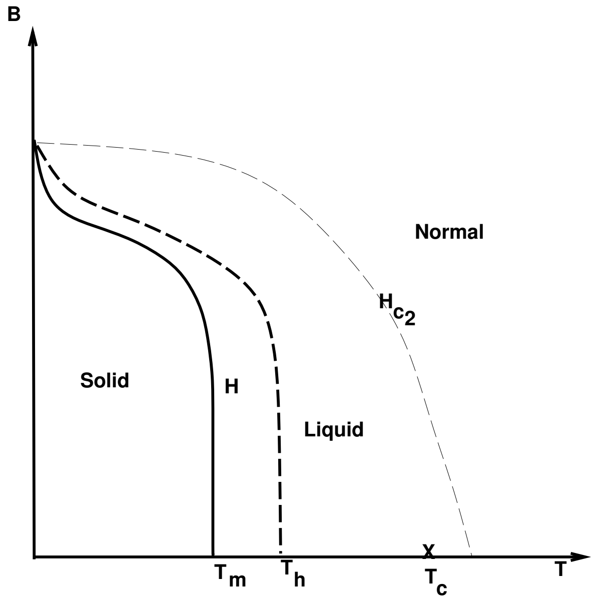

Figure 2: Phase diagram of a superconducting film in a perpendicular field within the KTBHN theory, according to ref. [37]. The phases are normal, vortex liquid, hexatic labelled H, and vortex solid. The line marked H_{c_2} is the mean field transition line, which, including fluctuation effects, is no longer a sharp transition but rather a crossover *region* between the vortex liquid and normal regions. The transition from solid to hexatic is mediated by unbinding of dislocation pairs, and that from hexatic to vortex liquid is mediated by unbinding of disclination pairs. Alternatively, the transition from solid to liquid might take place through a single first order transition, at a temperature below that of the solid hexatic transition in the KTBHN theory.

According to conventional flux creep theory [47, 48, 34], the system is then not a superconductor because vortices will move under the action of the Lorentz force arising from the current see Eq. (55), and the flow of vortices will give a voltage, see Eq. (53). The vortices are pinned by the disorder, so vortex motion is activated, and therefore very slow at low-T. Nonetheless, the linear resistivity is finite at all temperatures. Furthermore, according to conventional flux creep theory, the current will affect fluctuations in the system, and hence lead to non-linearities in the current-voltage characteristics, (*i.e.* deviations from Ohms law) when $f_L L_P \sim k_B T$ where f_L is the Lorentz force in Eq, (55) and L_P is the Larkin length, the range of vortex lattice translational order. In other words, deviations from Ohm's law will occur for $J \simeq J_{nl}$ where $J_{nl} \propto T$.

More recently, another picture has been proposed [11]. It is argued that, even though there is no ordering at any wavevector due to the effects of frustration and disorder, the system has a stiffness associated with it such that a perturbation applied in a localized region will affect the system out to a distance, ξ, called the vortex glass correlation length, which is presumably something like the Larkin length, L_P, at high temperatures but which can be much larger at low-T. The vortex glass correlation length might even diverge at some temperature T_c, below the system is in a vortex glass state, analogous to the spin glass state in magnetic systems [12], with vanishing linear resistivity. There is both experimental [49] and theoretical [50, 51] evidence that a vortex glass transition occurs at a finite T_c in three-dimensional systems. However, in two-dimensional systems, the vortex glass transition takes place at $T = 0$ [13, 14], just as in spin glasses. Nonetheless, there are experimental consequences of the $T = 0$ vortex glass transition since the correlation length diverges in this limit as $T^{-\nu}$. Hence deviations from Ohm's law set in when $f_L \xi \simeq k_B T$, which, from Eq. (55), gives the current scale for this to occur as

$$J_{nl} \propto T^{1+\nu} \quad , \tag{62}$$

different from the conventional flux creep prediction, $J_{nl} \propto T$. Recent experiments [14] on very thin ($16 \mathring{A}$) films of YBCO show that $J_{nl} \propto T^3$, implying a vortex glass correlation length exponent, $\nu \simeq 2.0$, in agreement with Monte Carlo simulations [13].

We conclude this lecture by describing how Monte Carlo simulations combined with finite-size scaling can be used to study vortex glass behavior in a simple two-dimensional model. The model that we study, known as the "gauge glass", has the following Hamiltonian:

$$\mathcal{H} = - \sum_{<i,j>} \cos(\phi_i - \phi_j - A_{ij}) \quad . \tag{63}$$

The phase, ϕ_i, is defined on each site of a square lattice, with periodic boundary conditions. The sum is over all nearest neighbor pairs on the lattice. The effects of the magnetic field and disorder are represented by the quenched vector potentials, A_{ij}, which we take to be independent random variables with a uniform distribution between 0 and 2π. This model seems to be the simplest one with the correct ingredients of randomness, frustration and order parameter symmetry. It does, however, ignore screening, which should be an excellent approximation in two-dimensions since the effective screening length, $\Lambda_{\rm eff}$ is very large, as discussed above.

If the A_{ij} are restricted to the values 0 and π, the model becomes the XY spin glass[12], for which the lower critical dimension is believed [52] to be 4. However, earlier work [51, 13, 50], has shown that the gauge glass is in a different universality class from the XY spin glass, presumably because it does not have the the "reflection" symmetry, $\phi_i \to -\phi_i \ \forall\, i$ [51]. As discussed in refs. [50, 13], it is useful to consider the change in free energy ΔF when one imposes a twist Θ along one of the space directions, x say. More precisely, the periodic boundary conditions, $\phi_i = \phi_{i+L\hat{x}}$ are replaced by the twisted boundary conditions, $\phi_i = \phi_{i+L\hat{x}} + \Theta$. By a simple redefinition of the phases ϕ_i one can replace this situation by a system with periodic boundary conditions and an extra contribution, Θ/L, to the vector potential on bonds in the x-direction.

By Monte Carlo methods one can calculate derivatives of the free energy w.r.t. Θ, so, for a single sample, we define a current, I, and a stiffness, Y, by

$$I \equiv \frac{\partial F}{\partial \Theta} = \frac{1}{L} \sum_i \langle \sin \Delta_i \rangle_T \quad , \tag{64}$$

$$Y \equiv \frac{\partial^2 F}{\partial \Theta^2} = \frac{1}{L^2} \Big\{ \sum_i \langle \cos \Delta_i \rangle_T - \ \frac{1}{T} \sum_{i,j} \Big[\langle \sin \Delta_i \sin \Delta_j \rangle_T - \ \langle \sin \Delta_i \rangle_T \langle \sin \Delta_j \rangle_T \Big] \Big\} \quad , \tag{65}$$

where $\Delta_i = \phi_i - \phi_{i+\hat{x}} - A_{i,i+\hat{x}}$, F is the total free energy and $i+\hat{x}$ refers to the nearest neighbor site in the x-direction from i. Note that both I and Y are gauge invariant so they are still useful even if one includes fluctuating gauge fields.

Above T_c, ΔF, and hence both I and Y, go to zero rapidly with increasing system size because the system is insensitive to boundary conditions when L is much greater than the vortex glass correlation length ξ. If T_c is finite, then, below T_c, I and Y vary with L as L^θ where θ (> 0), is an exponent describing the low temperature phase. In other words, I and Y *increase* with increasing L below T_c, the opposite of what happens above T_c. Precisely at T_c, both I and Y are independent of size. Hence if T_c is finite, I and Y should come together at T_c and splay out again at lower temperatures. By contrast, if $T_c = 0$, then, at $T = 0$, I and Y vary as L^θ but with $\theta < 0$. Consequently, I and Y decrease with L even at $T = 0$.

In a disordered system, it is necessary to perform an average over different realizations of the disorder, which we indicate by $[\cdots]_{\rm av}$. For the gauge glass, the average values of I and Y are both zero, i.e.

$$[Y]_{\rm av} = [I]_{\rm av} = 0 \quad , \tag{66}$$

because the configuration in which the vector potentials in the x direction have been increased by Θ/L has the same weight in the configurational average as the original

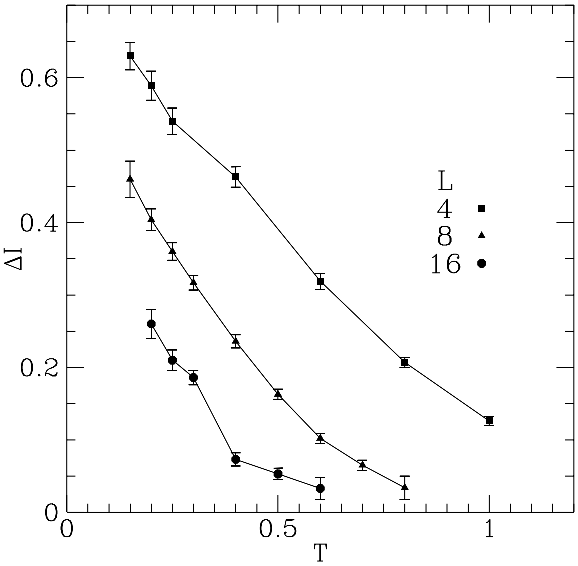

Figure 3: The r.m.s. current, $\Delta I = [I^2]_{av}^{1/2}$, for $d = 2$ determined by Monte Carlo simulations for different sizes and temperatures. The curves for different sizes do not come together, even at the lowest temperature. This behavior indicates a transition at $T = 0$.

choice of vector potentials. One is therefore interested in the root mean square fluctutation between samples. This means that many samples must be averaged over, typically several thousand. If T_c is finite, the finite size scaling form for the r.m.s. current, ΔI is therefore

$$\Delta I \equiv [I^2]_{\rm av}^{1/2} = \tilde{I}(L^{1/\nu}(T - T_c)) \quad (T_c > 0) \quad , \tag{67}$$

where ν is the correlation length exponent. We shall concentrate on the r.m.s. current in what follows, rather than the stiffness, because sample to sample flucutations in the stiffness have an asymmetric distribution with a long tail, which makes it difficult to get good statistics [50]. If $T_c = 0$ then ΔI decreases with size even at $T = 0$, i.e. $\Delta I \sim L^{\theta}$ where θ is negative and related to the exponent ν giving the divergence of the correlation length as $T \to 0$ by $-\theta = 1/\nu$. The finite size scaling form is then

$$L^{1/\nu} \Delta I = \tilde{I}(L^{1/\nu} T) \quad (T_c = 0) \quad . \tag{68}$$

Tests to ensure equilibration were carried out as described elsewhere [53].

The results for ΔI are shown in Fig. 3. Notice that they even at the lowest temperature, ΔI decreases with increasing size. This is precisely what is expected at a zero temperature transition, and the scaling plot in Fig. 4 corresponding to Eq. (68) works very well. From the fit we estimate

$$T_c = 0, \quad \nu = 2.2 \pm 0.2 \quad (d = 2) \quad . \tag{69}$$

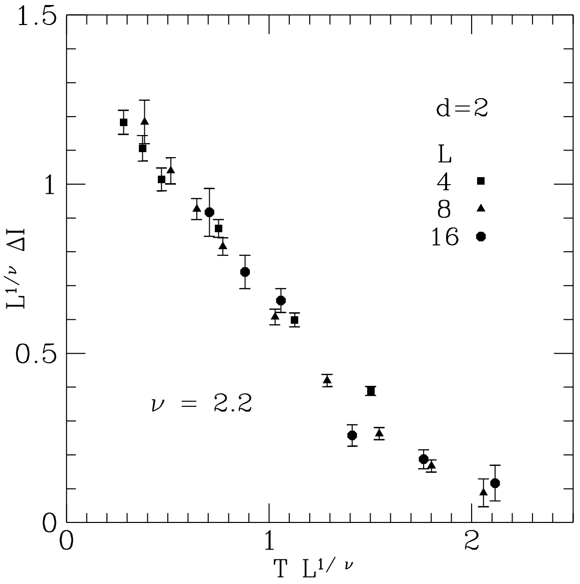

Figure 4: The same data as in Figure 3 but in a finite size scaling plot, with $T_c = 0$ and $\nu = 2.2$.

The value for ν agrees with earlier work [13], in which a different finite-size-scaling technique was used, and with recent experiments [14] on very thin (16Å) films of YBCO, in which current-voltage characteristics were determined.

6 Conclusions

In these lectures we have seen that two dimensional superfluids can have a finite-temperature phase transition, controlled by thermally excited vortices, with no condensate fraction in the low-T phase (*i.e.* there is no order parameter) but with a finite superfluid density. The KTB transition determines the nature of the transition in ^{4}He films and very thin superconducting films in the absence of a magnetic field. It may also describe the melting of the flux lattice in clean superconducting films subjected a perpendicular field, though it is also possible that this transition may be first order. Disordered superconducting films in a field show a vortex glass transition at $T = 0$, which has been seen experimentally.

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