

Physics 5I

The Twin Paradox Revisited

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(Dated: October 14, 2009)

In this handout we investigate the twin paradox of special relativity in rather more detail than is usually done.

The situation is illustrated in Fig. 1. To make things concrete we assume that the traveling twin, whom we call Alice, travels from earth a distance of $x = 4$ light years (as measured in the frame of the twin who stays on earth, called Bob) at a speed $v = \frac{4}{5}c$. She then turns round and comes back at the same speed. Hence, according to Bob, the journey takes a total of $t = 2 \times 4 / (\frac{4}{5}) = 10$ years. However, clocks in Alice's frame run slower by a factor of $\sqrt{1 - v^2/c^2} = 3/5$, because of time dilation. Hence she is only $t' = 10 \times \frac{3}{5} = 6$ years older when she returns to earth, whereas Bob (who stayed on earth) is 10 years older.

The apparent paradox is that looking from Alice's point of view, clocks on earth are running more slowly, and so it is Bob who should be younger at the end of the trip.

The resolution of the apparent paradox is that, to use special relativity, one needs to be in an inertial frame. Bob is in an inertial frame, so his conclusions are correct. Alice is not because she switches inertial frames when she begins the journey back.

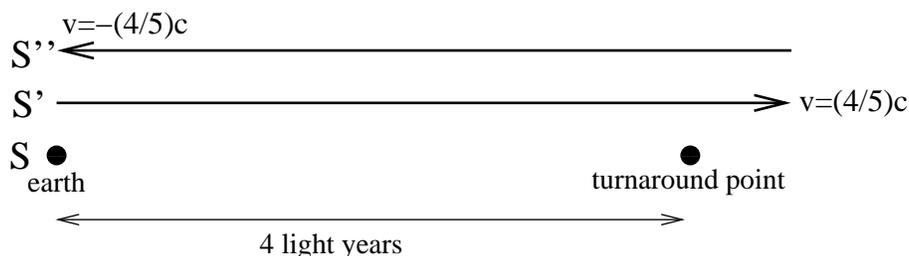


FIG. 1: In the twin “paradox”, one twin, Bob, stays on earth, an inertial frame which we will call S . At time zero, the other twin, Alice, gets on a moving walkway going with speed $v = \frac{4}{5}c$. This is also an inertial frame, which we call S' . She travels a distance 4 light years (as measured by Bob) and then jumps on to another moving walkway, moving in the opposite direction at the same speed, which carries her back to earth. This walkway is a third inertial frame, which we call S'' . We consider three events: event A when Alice leaves earth, event B when she turns round by jumping on to the returning walkway, and event C when she arrives back on earth.

In this handout we look more carefully at the consequences of this switch of inertial frames. We shall see that to analyze the problem correctly using special relativity we **need to consider three inertial frames**:

- Frame S is that of Bob and is fixed on earth. Distance and time are denoted by x and t .
- Frame S' , is that of Alice on her outbound journey. It is moving with respect to S at a velocity $+v$. Distance and time are denoted by x' and t' .
- Frame S'' , is that of Alice on her return journey. It is moving with respect to S at a velocity $-v$. Distance and time are denoted by x'' and t'' .

Relations between the distances and times in the three frames are given by the Lorentz transformations:

$$x' = \gamma(x - vt), \quad (1a)$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \quad (1b)$$

$$x'' = \gamma(x + vt), \quad (2a)$$

$$t'' = \gamma\left(t + \frac{v}{c^2}x\right), \quad (2b)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ ($= 5/3$ here). We have synchronized these three frames so $x = t = 0$ in frame S corresponds to $x' = t' = 0$ in frame S' and $x'' = t'' = 0$ in frame S'' . Bob stays in frame S . Alice is in frame S' until she turns around, at which point she “jumps” into frame S'' .

We denote the “event” of Alice leaving earth as event A, her arriving at the farthest point of her trip (where she turns around) as event B, and her returning to earth as event C. We now determine the position and time of each of these events in each of the three frames.

Event A. We already mentioned that we synchronized the frames such that

$$x_A = 0, \quad t_A = 0, \quad (S) \quad (3a)$$

$$x'_A = 0, \quad t'_A = 0, \quad (S') \quad (3b)$$

$$x''_A = 0, \quad t''_A = 0. \quad (S'') \quad (3c)$$

Event B. In frame S this occurs when $x_B = 4$ light years and, since $v = \frac{4}{5}c$, we have $t_B = 5$ years. Putting these values into Eqs. (1) and (2) we get

$$x_B = 4, \quad \boxed{t_B = 5}, \quad (S) \quad (4a)$$

$$x'_B = 0, \quad \boxed{t'_B = 3}, \quad (S') \quad (4b)$$

$$x''_B = \frac{40}{3}, \quad \boxed{t''_B = \frac{41}{3}}, \quad (S'') \quad (4c)$$

For example, to get the result for t'_B in Eq. (4b), we have, from Eq. (1b) with $x = 4, t = 5, v = \frac{4}{5}$, $t'_B = \frac{5}{3}(5 - \frac{4}{5}4) = 3$ years. Equation (4b) displays the expected time dilation, since the time is $t'_B = 3$ years for Alice, as opposed to $t_B = 5$ years for Bob.

More importantly, the time of event B (the turnaround) is greater in frame S'' than in frame S' by an amount $\frac{41}{3} - 3 = \frac{32}{3}$ years. Hence, when Alice jumps into frame S'' she should adjust her watch by this amount for it to be synchronized with her new frame. Of course, she has not aged by this additional amount, just as, for example, ones age does not change by a hour when crossing into a new time zone.

Event C. Event C happens when Alice returns to earth, i.e. $x_C = 0$ (the distance is zero in the rest-frame of the earth (S)). In frame S an extra 5 years have passed since event B , since Alice goes 4 light years at a speed of $\frac{4}{5}c$, so $t_C = 5 + 5 = 10$ years. Putting these values into Eqs. (1) and (2) we get

$$x_C = 0, \quad \boxed{t_C = 10}, \quad (S) \quad (5a)$$

$$x'_C = -\frac{40}{3}, \quad t'_C = \frac{50}{3}, \quad (S') \quad (5b)$$

$$x''_C = \frac{40}{3}, \quad \boxed{t''_C = \frac{50}{3}}, \quad (S'') \quad (5c)$$

Note that $t''_C - t''_B = \frac{50}{3} - \frac{41}{3} = 3$, so Alice has aged a further 3 years on the return trip as expected. Since she is in frame S' between events A and B, and in frame S'' between events B and C, she has aged by a total amount

$$t_{\text{Alice}} = (t'_B - t'_A) + (t''_C - t''_B) = 6 \text{ years}, \quad (6)$$

whereas Bob has aged by an amount

$$t_{\text{Bob}} = t_C - t_A = 10 \text{ years}. \quad (7)$$

The ratio of the Alice's age-change to that of Bob is therefore, as expected,

$$\boxed{\frac{t_{\text{Alice}}}{t_{\text{Bob}}} = \frac{3}{5} = \frac{1}{\gamma}}, \quad (8)$$

so **Alice (the traveling twin) is younger.**

However, if Alice adjusts her watch to synchronize with the new frame when jumping frames at the turnaround point, the time elapsed on her watch when she gets back to earth is $t''_C - t'_A = \frac{50}{3}$. Denoting this by $t_{\text{Alice's watch}}$ we have

$$\boxed{\frac{t_{\text{Alice's watch}}}{t_{\text{Bob}}} = \frac{5}{3} = \gamma,} \quad (9)$$

so **more time has elapsed on Alice's watch than on Bob's watch** by a ratio of $\frac{5}{3}$, which is the reciprocal of the ratio of the actual age-changes in Eq. (8). The difference between these ratios comes from the time difference of $\frac{32}{3}$ years between frames S' and S'' when Alice jumps between them (compare t'_B and t''_B in Eqs. (4b) and (4c)). It is noteworthy that the ratio, $\gamma = 5/3$, by which the time on Alice's *exceeds* that on Bob's watch is the *same* as the ratio by which the actual age-change of the traveling twin is *less than* that of the earth-bound twin.

You may also be interested to read the discussion of the twin paradox in Mermin's excellent book on special relativity, "It's about time", which I will put on reserve in the library. A traditional discussion, similar to that described here, is given in pages 119-123, but especially interesting is the discussion in Ch. 12 of how to discuss the problem from the point of view Alice (who has accelerations when she turns round) using Einstein's theory of *general relativity* (which includes gravity).