

PHYSICS 250

Homework 8

Due in the last class, Friday December 7.

1. *Bookwork*

For a differential equation of the Sturm-Liouville type show that

- (a) The eigenvalues are real.
- (b) The eigenfunctions are orthogonal.

2. *Variational Problem*

You are familiar with the solution of the simple harmonic oscillator problem in quantum mechanics with Hamiltonian

$$\mathcal{H} = -\frac{1}{2} \frac{d^2\psi(x)}{dx^2} + \frac{1}{2} x^2\psi(x) = E\psi(x).$$

Note that we have set $\hbar = m = \omega = 1$ where m is the mass and ω the classical frequency of oscillation.

The ground state wavefunction $\psi_0(x)$ is actually a Gaussian, but let's suppose you don't know that. Suppose instead you guess that the groundstate wavefunction is actually an exponential

$$\psi(x) = Ae^{-\kappa|x|},$$

where κ is a parameter and A is a normalization constant, which you should determine,

Estimate the ground state energy by the variational approach, in which you vary κ .

Compare your answer with the exact result.

3. Kelly, Qu. 8, Ch. 7.

4. Kelly, Qu. 13, Ch. 7.

Note: For part (a) a precise discuss of the meaning of "self adjoint" is given in Sec. 7.3.1.

5. Consider the probability distribution

$$P_1(x) = \begin{cases} 2(1-x), & (0 < x < 1), \\ 0, & (\text{otherwise}). \end{cases}$$

- (a) Determine the mean and standard deviation of P_1 .
- (b) The skewness, s , of a distribution is defined by

$$s = \langle (x - \langle x \rangle)^3 \rangle / \sigma^3,$$

where σ^2 is the variance. Find the skewness for $P_1(x)$ and also for the distribution

$$P_2(x) = \begin{cases} 1, & (0 < x < 1), \\ 0, & (\text{otherwise}). \end{cases}$$

Note that the skewness vanishes for a distribution, like P_2 , which is symmetric about its mean.

6. A radioactive object has, *on average* 5 decays per second. What is the probability of detecting

- (a) 0

- (b) 1
- (c) 5
- (d) 20

decays in a second.

7. For a Gaussian distribution show that the probability that a result is obtained within $k\sigma$ from the mean is $\text{erf}(k/\sqrt{2})$ where $\text{erf}(x)$ is the error function.
8. A casino claims that there is a 50/50 chance of winning at a particular game. I play 100 times and lose 70 times.
 - (a) Show that this corresponds to 4 times the standard deviation if the probability of winning is really 50%.
 - (b) We showed in class that the distribution for this problem (the binomial distribution) goes over to a Gaussian distribution for a large number of trials. Since 100 is quite a large number we assume that this is a good approximation here. From the previous question, we see that the probability of a $\pm 4\sigma$ deviation, or greater, is $\text{erfc}(4/\sqrt{2})$. You are given that $\text{erfc}(4/\sqrt{2}) = 0.00006334$. Do you think that the casino was being honest in claiming a 50/50 chance of winning?
9. *An example where the central limit theorem doesn't hold*
Consider the Lorentzian distribution

$$P(x) = \frac{1}{\pi} \frac{1}{1+x^2}.$$

Note that the standard deviation does not exist (and hence the central limit theorem does not apply.)

- (a) By computing the Fourier transform, as discussed in class, obtain the distribution of the sum of N random variables with a Lorentzian distribution
 - (b) Hence show that the distribution of the mean of N variables is *precisely the original Lorentzian distribution*.
Hence, in contrast to cases where the central limit theorem applies, the distribution of the sum does *not* become sharper than the distribution of an individual variable.
10. I carry out the following series of measurements for a particular quantity, x , which is subject to random noise:

$$1.1, 0.9, 0.95, 1.05, 1.0.$$

What is my best estimate for $\langle x \rangle$ and what is the error bar on this estimate?