PHYSICS 115/242

Homework 5

Due in class, Monday, May 12.

1. *Central Limit Theorem*

   (a) With x_i a random number with a uniform distribution between 0 and 1 show analytically that the distribution of X, where

   $$X = \sqrt{\frac{12}{N}} \sum_{i=1}^{N} (x_i - 1/2),$$

   has zero mean and variance unity.

   (b) Choosing $N = 12$, verify numerically that X has (approximately) a *Gaussian* distribution by computing the first 6 moments for a sufficiently large sample,
   i.e. Generate a large number of values for X and show that for your sample $\langle X \rangle \simeq \langle X^3 \rangle \simeq \langle X^5 \rangle \simeq 0, \langle X^2 \rangle \simeq 1, \langle X^4 \rangle \simeq 3, \langle X^6 \rangle \simeq 15$.
   Comment: You will find that $\langle x^6 \rangle$ is somewhat too low (because the tails of the distribution are not well represented). This could be rectified by using a larger value for N. The value of $\langle x^4 \rangle$ is also a slightly low, though the effect is less than for $\langle x^6 \rangle$. (See the comments in the next part.)

   (c) *242 students only.*
   Show that, for general N, $\langle X^4 \rangle = 3 - 6/(5N)$ and $\langle X^6 \rangle = 15 - 18/N + 48/(7N^2)$. For $N = 12$ this gives $\langle X^4 \rangle = 2.9$, rather than 3, and $\langle X^6 \rangle = 13.5476$, rather than 15. Compare these values with your numerical results.

2. *Lorentzian distribution*

   (a) Explain how to generate random numbers with a Lorentzian distribution

   $$P(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \qquad (-\infty < x < \infty).$$

   (b) Generate a (fairly large) sample of numbers with this distribution (don't print them out!) and calculate what fraction lie in the range $|x| < 1$. Compare your numerical result with the exact answer.

   (c) In part (b) I didn't ask you to calculate moments. Why not?
   Note: Related to the answer to this question, I point out that the Lorentzian is one distribution for which the central limit theorem does *not* hold.

3. *Monte Carlo Integration with error bars*

   (a) Evaluate the following integral

   $$\int_1^2 \ln x \, dx$$

   using Monte Carlo integration. You *must* give an error bar (obtained from the computation) for your answer. Do one run with a large number of points (N), and from this *one* set of data estimate the error bar in your answer.

(b) The error bar is one standard deviation. If we assume that you have enough points for the central limit theory to apply, so the distribution of the sample mean is Gaussian, there is a probability of 68% that the exact result lies in within σ of the exact answer, 95.5% probability within $2\,\sigma$, 99.7% probability within $3\,\sigma$, etc. (where in this context the symbol σ refers generically to the standard deviation).

Evaluate the integral analytically, compare your result from Monte Carlo integration with the exact result, and comment.

4. *Multi-dimensional Monte Carlo Integral*
Consider the following 10-dimensional integral

$$I = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_{10}\; (x_1 + x_2 + \cdots + x_{10})^2 \,.$$

(a) Show that the exact answer is 155/6.
Hint: expand the square and integrate each term.

(b) Estimate the answer numerically using a Monte Carlo method. Obtain the error bar on your estimate and compare with the exact answer.

5. *Random Walk*
Consider a random walker who starts at $x = 0$ and walks along a line along the x-axis. At each time step, $t = 1, 2, 3, \cdots$, the walker moves one step to the right or one step to the left with equal probability. By averaging over a sufficiently large number of walks show numerically that

$$\langle x(t) \rangle \simeq 0; \quad \langle x^2(t) \rangle \simeq t$$

where the average $\langle \cdots \rangle$ is over your sample of walks. You should plot (or produce a neat table of) $\langle x(t) \rangle$ and $\langle x^2(t) \rangle$ against t for a range of t.