PHYSICS 242

FINAL EXAM, 2014

Due in my mailbox in ISB 232 by 5 pm. on Tuesday, June 10.

This must be your own work. **No collaboration is allowed.** You may use your notes, books, and the web.

For the C/C^{++}/Fortran/Java questions you must write a program in C, C^{++}, Java or Fortran, but not use Mathematica. (Of course, nothing stops you from *checking* your answer with Mathematica.) You must provide a listing of the program, the output, and any relevant explanation. Group all parts of a question together.
For the Mathematica questions you must provide a listing of the commands you used, the output, and any relevant explanation. Put all your answers to the Mathematica part in one notebook.

C/C^{++}/Fortran/Java part

1. Find the best straight-line fit to the following data:

```
#      x          y       error in y
     1.1075     5.1807     0.0813
     2.0198     7.0209     0.0811
     2.8948     8.7185     0.0487
     3.8009    10.5185     0.0736
     4.8840    12.7217     0.0621
     6.1507    15.3488     0.1496
     7.0411    17.1824     0.0853
     8.0592    19.1125     0.1110
     8.8211    20.7522     0.1253
    10.0766    23.5099     0.1433
```

   (Download this data from `http://young.physics.ucsc.edu/115/homework/qu1fitdata`.)

   You need to determine the intercept and slope and the error bars on these quantities.

2. Evaluate

$$I = \int_0^2 e^{-x^3}\, dx$$

   numerically to 13 decimal places.
   Note: You must explain how you estimated that the desired accuracy had been obtained.

3. In my research, I recently needed to find the solution for T of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^2/2) \tanh^2\left(\frac{x}{T}\right) dx = \frac{1}{2}.$$

   Solve this equation for T to 4 decimal places.
   Note: To integrate to $\pm\infty$ you may (i) integrate out to some large but finite value and forget about the rest, or (ii) perform a change of variables to make the range of integration finite, or (iii) use the Gaussian quadrature methods that we mentioned briefly in class. To do the root-finding I suggest that you use bisection (safe and sure).
   Note: Please discuss how you estimated that the desired error had been obtained.

4. Consider the differential equation

$$\frac{dy}{dx} = \sqrt{y + x^2}.$$

   Find the value of $y(2)$ correct to 10 decimal places given that $y(0) = 1$.
   Note: You must explain how you estimated that the desired accuracy had been obtained.

5. Compute the following integral by Monte Carlo methods:

$$I = \int_0^2 dx_1 \int_0^2 dx_2 \cdots \int_0^2 dx_8 \, \frac{1}{1 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8} \, .$$

You must indicate how many points you generated and give an estimate for the error bar.

Mathematica part

6. (a) Find numerically all the solutions of

$$x^8 - 4x^3 = 3 \, .$$

   (b) Determine how many integers between 10^6 and 1.5×10^6 are perfect squares, (e.g. like 64 which is equal to 8^2.)

   (c) Find analytically (using Mathematica) the value of the following integral

$$\int_0^\infty \frac{1}{(1 + x^n)^m} \, dx \, ,$$

   assuming $n > 0, m > 0, nm > 1$. Find a simpler expression, not involving Gamma functions, for the case of $m = 1$.

7. In appropriate units, mean field theory predicts that the magnetization m of an Ising magnet *in a magnetic field* h is given by

$$m = \tanh\left(\frac{Jm + h}{T}\right) \, .$$

You are given that the physical solution is the one with $m > 0$. Set the interaction J to be unity.

   (a) What is m at $T = 1$ for $h = 0.001$?

   (b) Still with $h = 0.001$, use a one-line command to plot m versus T, with T ranging from a value close to zero up to $T = 1.3$.

   (c) Do a log-log plot of m against h at $T = 1$ (which is the zero-field transition temperature, though you are not required to show that here) for h between 10^{-4} and 10^{-1}.
   Note: There is a Mathematica command `LogLogPlot`.

   (d) One can easily show analytically that the correct result for small h in the previous part is $m = (3h)^{1/3}$. Verify this by plotting $(3h)^{1/3}$ and m on the same log-log plot, for the same range as in the previous part.

8. Consider the "sine map"

$$x_{n+1} = \lambda \sin(\pi x_n),$$

where λ is a parameter in the range $0 < \lambda \leq 1$, and the x_n lie in the range from 0 to 1.

   (a) Find *analytically* the value of λ at which the fixed point at $x = 0$ becomes unstable.

   (b) Show that the value of λ at which the fixed point at non-zero x becomes unstable can be expressed as the solution of two coupled equations, for λ and $x^\star$, the fixed point value. Solve these equations numerically.

   (c) For each of the following values of λ: 0.6, 0.8, 0.85, 0.89, 0.94 and 1, determine the Lyapunov exponent, λ_L, and state whether the trajectory is chaotic or not.

   (d) For the cases of non-chaotic behavior, determine graphically or otherwise (for example by using the Mathematica function `FixedPoint`) whether one has a fixed point or limit cycle. If you find a limit cycle, determine its length.

(e) For $\lambda = 0.89$ and $x_0 = 1/5$, compute x_1, x_2 and x_{10^4}.

9. Consider a quantum mechanical particle in the anharmonic potential

$$\mathcal{H} = \frac{p^2}{2} + \frac{x^2}{2} + x^4.$$

(a) Starting from Schrödinger's equation, use the "shooting method" discussed in class to determine the lowest three energy levels.

(b) Also determine the first *ten* energy levels using the *matrix* methods discussed in class, and compare your answer for the first three with the results in part (a).
Note: You must explain how you tested for convergence.

Note: Take $\hbar = m = 1$.