PHYSICS 232

Homework 5

Due in my mailbox, Friday March 17

As usual, answering these howework questions is important for your understanding of the subject. If you have difficulty with them, and *especially if you don't know where to start*, come and see me.

1. **Thermodynamics of the Superconducting State**
   Consider a superconductor in the shape of a long cylinder parallel the field so demagnetization effects can be neglected, *i.e.* the value of $\mathbf{H}$ inside (and outside) the sample is equal to $\mathbf{B}_{\text{ext}}$. Assume that the sample has unit volume. The Ginzburg-Landau free energy discussed in class satisfies the following thermodynamic identity

$$dF = -S\,dT - M\,dH\,,$$

   where S is the entropy and M the magnetization. dH should really be dB_{ext} (see *e.g.* the paper by Narayan and Young) but these are equal here so we use the common notation dH. The phase boundary between the superconducting and normal states is at the critical field $H_c(T)$. (We assume a type-I superconductor here.)

   (a) Use the fact that F is continuous across the phase boundary to show that

$$\frac{dH_c(T)}{dT} = \frac{S_n - S_s}{M_s - M_n}\,,$$

   where the subscripts s and n indicate values in the superconducting and normal states. This is the analogue of the Clausius-Clapeyron equation for solid-liquid melting.

   (b) Since the superconducting state has perfect diamagnetism, so $B_s = 0$, and the normal state is only very weakly magnetic, so $M_n \simeq 0$, show that the entropy discontinuity across the phase boundary is

$$S_n - S_s = -\frac{H_c}{4\pi}\frac{dH_c(T)}{dT}\,, \tag{1}$$

   and hence that the latent heat at the transition *in a field* is

$$Q = -T\,\frac{H_c}{4\pi}\frac{dH_c(T)}{dT}\,.$$

   Note that this tends to zero as the field tends to zero since $dH_c(T)/dT$ is finite (it is also *negative*).

   (c) Show that when the transition occurs in *zero field* there is a discontinuity, not in the entropy, but in the specific heat which is given by

$$C_s - C_n = \frac{T}{4\pi}\left(\frac{dH_c(T)}{dT}\right)^2\,.$$

2. **The London Equation for Superconducting Slab**
Consider an infinite superconducting slab bounded by two parallel planes perpendicular to the y axis at $y = \pm d$. There is a uniform magnetic field of strength H_0 ($= B_{\text{ext}}$) applied along the z axis.

   (a) The boundary condition is that $\mathbf{B}$ is continuous at the boundary. Note that $\mathbf{B}$ is tangential to the surface and the more familiar boundary condition is that the tangential component of $\mathbf{H}$ is continuous. However, this applies to the situation when there is a surface current in an *infinitesimally thin* layer whose value we don't want to calculate explicitly. Here we are explicitly including, in the calculation, the current in the material (which is *not* in an infinitesimally thin region at the surface). Hence the true magnetic field $\mathbf{B}$ should be continuous.
   Deduce from the London equation

$$\boldsymbol{\nabla} \times \mathbf{J} = -\frac{c}{4\pi\lambda_L^2}\mathbf{B}$$

   and the Maxwell equation

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}\,,$$

   that within the superconductor

$$\mathbf{B} = B(y)\,\hat{\mathbf{z}}, \qquad \text{where } B(y) = H_0\,\frac{\cosh(y/\lambda_L)}{\cosh(d/\lambda_L)}\,,$$

   where

$$\lambda_L = \left(\frac{4\pi n_s e^2}{mc^2}\right)^{1/2}$$

   is the London penetration depth.

   (b) Show that the diamagnetic current density flowing is

$$\mathbf{J} = J(y)\,\hat{\mathbf{z}}, \qquad \text{where } j(y) = \frac{c}{4\pi\lambda_L}H_0\,\frac{\sinh(y/\lambda_L)}{\cosh(d/\lambda_L)}\,,$$

   (c) The magnetization density at a point within the slab is given by $\mathbf{M}(y) = (\mathbf{B}(y) = \mathbf{H}_0)/4\pi$. Show that the magnetization density averaged over the thickness of the slab is given by

$$\overline{M_z} = -\frac{H_0}{4\pi}\left(1 - \frac{\lambda_L}{d}\tanh\frac{d}{\lambda_L}\right)\,.$$

   Give the limiting forms for the susceptibility χ, defined by $\overline{M_z} = \chi H_0$, when the slab is thick ($d \gg \lambda_L$) and thin ($d \ll \lambda_L$).

3. **The critical field $\mathbf{H_{c_2}}$**
In this question we determine the upper critical field H_{c_2} of a type-II superconductor. This transition turns out to be continuous (second order) at least in Ginzburg-Landau theory,

in which case one can locate the transition by looking only at the quadratic terms. These are

$$F = \int dV \left[\alpha(T)|\psi|^2 + \frac{\hbar^2}{2m^\star} \left| \left(-i\boldsymbol{\nabla} - \frac{e^\star \mathbf{A}}{\hbar c} \right) \psi \right|^2 \right] .$$

First of all, take $H = 0$, look for a uniform solution, and minimize with respect to $\psi^\star$. This gives

$$\alpha(T)\, \psi = 0 \,,$$

showing that the zero field transition occurs when $\alpha(T) = 0$, as discussed in class.

However, at H_{c_2}, the order parameter is non-uniform, since it describes the flux-line lattice, and so the minimization condition becomes

$$-\frac{\hbar^2}{2m^\star} \left(\boldsymbol{\nabla} - i\frac{e^\star \mathbf{A}}{\hbar c} \right)^2 \psi - |\alpha(T)|\, \psi = 0 \,. \tag{2}$$

We look for the point where a solution of this equation with non-zero $\psi(\mathbf{r})$ first appears, which is an *eigenvalue problem*. In effect, we are looking for a *zero eigenvalue* of the operator which acts on $\psi(\mathbf{r})$ in Eq. (2).

(a) Since we consider only the onset of ordering, the magnetic field inside the sample will be the same as that outside, namely $\mathbf{H}$. Hence we write $\mathbf{B} \equiv \boldsymbol{\nabla} \times \mathbf{A} = \mathbf{H}$ and take the gauge where

$$\mathbf{A} = H\, \hat{\mathbf{y}}\, x \,,$$

which corresponds to $\mathbf{H}$ in the z-direction. Look for a solution of the form

$$\psi = \exp(ik_y y + ik_z z)\, u(x) \,,$$

and show that $u(x)$ satisfies the equation

$$\frac{\hbar^2}{2m} \left\{ -u''(x) + \left[k_z^2 + \left(k_y - \frac{Hxe^\star}{c\hbar} \right) \right] u(x) \right\} = |\alpha(T)|\, u(x) \,.$$

This is the same as the problem of a quantum mechanical particle in a magnetic field.

(b) The first instability will be for $k_z = 0$, and k_y can be eliminated by a shift in x. Hence we are looking for a solution of

$$-\frac{\hbar^2}{2m} u''(x) + \frac{e^\star H}{2m^\star c}^{\,2}\, x^2\, u(x) = |\alpha(T)|\, u(x) \,.$$

Comparing this with the simple harmonic oscillator, show that a solution first appears (on increasing H) at $H = H_{c_2}$ where H_{c_2} is given by

$$|\alpha(T)| = \frac{1}{2} \frac{e^\star H_{c_2}}{m^\star c} \,.$$

(c) Relating $\alpha(T)$ to ξ^2, as discussed in class and in Qu. 4 below, show that

$$H_{c_2} = \frac{\phi_0}{2\pi\xi^2}\,, \tag{3}$$

where $\phi_0 = hc/2e$ is the flux quantum for a charge $e^\star = 2e$.

Note: Eq. (3) has a simple interpretation. Since there is a flux ϕ_0 per flux line, it tells us that ordering disappears when the separation between flux lines in the flux line-lattice phase becomes equal to about ξ. Now the center of a flux line is normal, and the order parameter "heals" to its equilibrium value within a distance ξ. Hence it is reasonable that the order parameter is zero everywhere when the flux lines get closer than about ξ.

4. **The size of the "critical region" of a superconductor.**
We are interested to know how close to the transition temperature, T_c, one needs to be to observe deviations from mean field behavior. In this question we estimate this using a criterion due to Ginzburg. The Ginzburg criterion is that critical fluctuations become important when the (mean field) ordering free energy in a volume whose linear dimension is the correlation length, ξ, becomes as small as k_BT, i.e. in d-dimensions:

$$\Delta F\,\xi^d \simeq k_BT_c.$$

Note: This should at least be plausible; you are **not** required to justify it.

Consider the the Ginzburg–Landau free energy,

$$F = \int\, dV\, \left[\alpha(T)|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m^\star}\left|\left(-i\boldsymbol{\nabla} - \frac{e^\star\mathbf{A}}{\hbar c}\right)\psi\right|^2\right],$$

with $e^\star = 2e$. We will look for minima of this functional (rather than integrating over all configurations). It turns out that this corresponds to doing a mean field theory, which will be sufficient to estimate where deviations from mean field theory *start* to appear.

(a) Consider zero field and look for a solution which is constant in space. Show that the ordering free energy per unit volume is

$$\Delta F = \frac{\alpha^2}{2\beta}\,.$$

and hence that the Ginzburg criterion in three dimensions is

$$\frac{\alpha^2\xi^3}{2\beta} \simeq k_BT.$$

(b) Show that, for $T > T_c$,

$$\xi(T) = \left(\frac{\hbar^2}{2m^\star|\alpha|}\right)^{1/2}$$

and so diverges as $(T - T_c)^{-1/2}$ You can then *define* $\xi(T)$ below T_c to be symmetrical about T_c, i.e. $\xi(T_c - \Delta T) = \xi(T_c + \Delta T)$.

(c) Show that for $T < T_c$, the penetration depth $\lambda(T)$ diverges as $(T_c - T)^{-1/2}$.

(d) Show that for $T < T_c$ the Ginzburg criterion translates to

$$\lambda(T) = \text{constant}\frac{\Lambda_T}{\kappa}\,,$$

where

$$\Lambda_T = \frac{\phi_0^2}{16\pi^2 k_B T}\,,$$

in which $\phi_0 = hc/2e$ is the the flux quantum, and $\kappa = \lambda(T)/\xi(T)$.

(e) Show that, if Λ_T is measured in Å and T in Kelvin, then

$$\Lambda_T \simeq \frac{2 \times 10^8}{T}\,.$$

(f) Hence, writing $\lambda = \lambda_0(1 - T/T_c)^{-1/2}$, where λ_0 is approximately the zero temperature limit of the penetration depth, show that the width of the critical region, ΔT, is given by

$$\frac{\Delta T}{T_c} \sim \left(\frac{\kappa\,\lambda_0\,T_c}{2 \times 10^8}\right)^2\,, \tag{4}$$

where λ_0 is in Å and T in Kelvin.
i.e. the width of the critical region *is generally tiny.* For example for $\kappa = 1, \lambda_0 = 100\text{Å}, T_c = 2\,K$, we get $\Delta T/T_c \simeq 10^{-12}$. The width of the critical region is larger in high-T_c materials because (i) T_c is larger, (ii) κ is large (the materials are strongly type-II), (iii) λ_0 is somewhat bigger, and (iv) (not considered in this question) high-T_c materials are strongly *anisotropic*, which turns out to also increase the size of the critical region.

(g) Show that for $d > 4$

$$\Delta F\,\xi^d \gg k_B T_c$$

close to T_c, indicating that the system never enters a region where fluctuations invalidate mean theory. It is true, rather generally, that mean field exponents (though not the mean field value of T_c) are exact for $d > 4$.

(h) Consider the Ginzburg criterion in *two-dimensions* and derive an expression analogous to Eq. (4). Show that the critical region is larger in two dimensions than in three.
Note: It is a general result that fluctuation effects, and the size of the critical region, are larger in lower dimensions.

5. **The Cooper Problem**
Consider electrons in a singlet state, described by the symmetric spatial wavefunction

$$\phi(\mathbf{r} - \mathbf{r}') = \int \frac{d^3k}{(2\pi)^3}\,\chi(\mathbf{k})\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,.$$

In the momentum representation, *i.e.* using a basis of plane wave states, the Schrödinger equation has the form

$$\left(E - 2\frac{\hbar^2 k^2}{2m}\right) = \int \frac{d^3k}{(2\pi)^3}\, V(\mathbf{k},\mathbf{k}')\,\chi(\mathbf{k}')\,,$$

where $V(\mathbf{k},\mathbf{k}')$ describes the interaction potential between the electrons.

Assume that the two electrons interact in the presence of a degenerate electron gas, whose only effect is to prevent, because of the exclusion principle, the two electons being considered accessing states with $k < k_F$, *i.e.*

$$\chi(\mathbf{k}) = 0, \qquad (k < k_F).$$

Note: Mathematically this means that the density of states at the lowest accessible level is finite, which is equivalent to the problem of a single electron in the absence of a degenerate gas *in two dimensions.*
We take the interaction potential to have the simple attractive form

$$V(\mathbf{k},\mathbf{k}') = \begin{cases} -V, & \epsilon_F \le \dfrac{\hbar^2 k_i^2}{2m} \le \epsilon_F + \hbar\omega. \quad (i = 1,2), \\ 0 & \text{(otherwise)}\,. \end{cases}$$

We look for a bound state solution, *i.e.* one with energy less than 2Δ, and define the binding energy to be

$$\Delta = 2\epsilon_F - E.$$

(a) Show that a bound state of energy E exists provided

$$1 = V \int_{\epsilon_F}^{\epsilon_F+\hbar\omega} \frac{N(\epsilon)\,d\epsilon}{2\epsilon - E}\,,$$

where $N(\epsilon)$ is the density of one-electron states *of a single spin.*

(b) Show that there is a solution with $E < 2\epsilon_F$ for arbitrarily weak V, provided that $N(\epsilon_F) \neq 0$.
Note the crucial role played by the exclusion principle. If the lower cutoff were not ϵ_F but 0 then there would not be solution for arbitrarily weak couplings in three dimensions because $N(0) = 0$.

(c) Assuming that $N(\epsilon)$ differs negligibly from the $N(\epsilon_F)$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega$, show that the binding energy is

$$\Delta = 2\hbar\omega\, \frac{e^{-2/N(\epsilon_F)V}}{1 - e^{-2/N(\epsilon_F)V}},$$

which becomes, in the weak coupling limit $N(\epsilon_F)V \ll 1$,

$$\Delta = 2\hbar\omega\, e^{-2/N(\epsilon_F)V}.$$

Note: the coefficient of $1/N(\epsilon_F)V$ in the exponential is twice what is found in the BCS solution (in which the electrons in the degenerate gas *also* participate in binding). Hence, in the strongly weak coupling limit, the energy gap and hence the transition temperature found in BCS is much larger than that which would naively be expected from the Cooper problem.

6. **Bound states in dimensions 1 to 3**
Consider a particle of mass m moving in a potential $V(r)$, given by

$$V(r) = -V_0 \ (r < a)$$

$$V(r) = 0 \ (r > a)$$

Compute the bound state energy for the case of dimension d equal to one. Also compute the critical value of V_0 needed for a bound state to form when $d = 3$.

(Optional) Find the binding energy in $d = 2$. (Note that the method is the same as for $d = 1, 3$ but is a little harder because the wavefunction involves Bessel functions rather than trigonometrical functions or exponentials.)