Physics 231 Homework 6, Solutions

1/ (a) There are 4 states per impurity level

energy	# of electrons
0	0
ε_d	1
ε_d	1
$2\varepsilon_d+\Delta$	2

Hence
$$\frac{n_d}{N_d} = \frac{0 + 2e^{-\beta(\varepsilon_d-\mu)} + 2e^{-\beta(2\varepsilon_d-2\mu+\Delta)}}{1 + 2e^{-\beta(\varepsilon_d-\mu)} + e^{-\beta(2\varepsilon_d-2\mu+\Delta)}}$$
$$= \frac{1 + e^{-\beta(\varepsilon_d-\mu+\Delta)}}{1+\frac{1}{2}e^{\beta(\varepsilon_d-\mu)}+\frac{1}{2}e^{-\beta(\varepsilon_d-\mu+\Delta)}}$$

(b) For $\Delta \to 0$
$$\frac{n_d}{N_d} = \frac{1+e^{-\beta(\varepsilon_d-\mu)}}{1+\frac{1}{2}e^{\beta(\varepsilon_d-\mu)}+\frac{1}{2}e^{-\beta(\varepsilon_d-\mu)}}$$
$$= \frac{2\left(1+e^{-\beta(\varepsilon_d-\mu)}\right)}{\left[1+e^{\beta(\varepsilon_d-\mu)}\right]\left[1+e^{-\beta(\varepsilon_d-\mu)}\right]} = \frac{2}{e^{\beta(\varepsilon_d-\mu)}+1}$$

i.e. twice the Fermi function as expected.

For $\Delta \to \infty$
$$\frac{n_d}{N_d} = \frac{1}{1+\frac{1}{2}e^{\beta(\varepsilon_d-\mu)}} \qquad \text{AM Eq. (28.32)}$$

(c) Assume that only 1 electron is present in any of the orbits. Then the states are:

ε_i (1 electron)

or 0 electrons

Hence
$$\frac{n_d}{N_d} = \frac{2\sum_i e^{-\beta(\varepsilon_i-\mu)}}{1+2\sum_i e^{-\beta(\varepsilon_i-\mu)}}$$

factors of 2 because there are 2 states with energy ε_i because of spin degeneracy.

$$= \frac{1}{1+\frac{1}{2}\left(\sum_i e^{-\beta(\varepsilon_i-\mu)}\right)^{-1}}$$

1 (d) Very little will change. Except at low T the impurities are fully ionized. This remains the same. At low T the fraction of ionized impurities is exponentially small. This remains qualitatively the same.

2 (a) $\varepsilon = 18$, $\frac{m_c}{m} = 0.015$

$$\varepsilon_d = \left(\frac{m_c}{m}\right) \frac{1}{\varepsilon^2} \times 13.6\,\text{eV} = 6.3 \times 10^{-4}\,\text{eV}$$

(b) $r_0 = \frac{m}{m_c}\,\varepsilon \times 0.529\,\text{Å} = 635\,\text{Å}$

(c) Need mean spacing to be about r_0

i.e. $\frac{1}{N_d} \simeq \frac{4\pi}{3} r_0^3 \Rightarrow N_d \simeq 10^{15}/\text{cm}^3$

3 (a) In Si there are 6 pockets along the <100> directions. The field is in a direction

$$\left(\frac{\sin\varphi}{\sqrt{2}}, \frac{\sin\varphi}{\sqrt{2}}, \cos\varphi\right) \quad \text{with } \varphi = 30^\circ$$

The cyclotron effective mass is the geometric mean of the band masses in directions $\perp$ to the field, see Qu. 1 of HW. 5.

From Kittel Ch. 8 (Eq. 34 in my edition) this leads to

$$\left(\frac{1}{m^*}\right)^2 = \frac{\cos^2\theta}{m_t^2} + \frac{\sin^2\theta}{m_l m_t}$$

where θ is the angle between the field and the direction of the major axis of the ellipsoid of constant energy, m_l is the longitudinal effective mass, and m_t is the transverse effective mass.

For the pockets along $(0,0,1)$ and $(1,0,0)$ $\theta = \varphi = 30^\circ$

For the pockets along $(\pm1,0,0)$ and $(0,\pm1,0)$

$$\cos\theta = \frac{1}{\sqrt{2}} \sin\varphi = \frac{1}{\sqrt{2}}\,\frac{1}{2} \quad \text{so } \theta = \cancel{52.2^\circ}\ 69.3^\circ$$

Because there are 2 values of θ there are 2 values for the resonance frequency.

③

(b) ~~With $m_c = m$,~~

$$\nu = \frac{\omega}{2\pi} = \frac{eH}{2\pi m_c c} = \frac{\cancel{9.1\times10^{-28}}\, 4.8\times10^{-10}}{6.28\times 9.1\times10^{-28}\times3\times10^{10}}\, H\, \frac{m}{m_c}$$

$$= 2.8\times10^{6} H \frac{m}{m_c}$$

if $\nu = 2.4\times10^{10}$

$$H = \frac{m_c}{m}\times\frac{2.4\times10^{10}}{2.8\times10^{6}} = 0.86\times10^{4}\frac{m_c}{m} \text{ Gauss}$$

With $m_L \simeq m$, $m_t \simeq 0.2 m$, $\theta = 30°$

$$\frac{m^2}{m_c^2} = \frac{3}{4}\frac{1}{0.2^2} + \frac{1}{4}\frac{1}{0.2} = 20.$$

$\frac{m_c}{m} = 0.22$, $\Rightarrow H = 8.6\times10^{3}\times 0.22 = 1.88\times10^{3}$ Gauss

This is the lower electron peak in Fig 28.9.

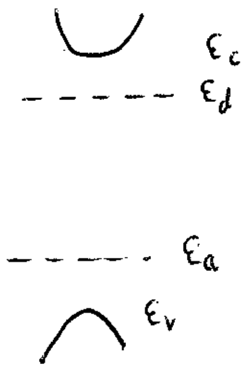
With $m_c \simeq m$, $m_t \simeq 0.2m$, $\theta \simeq 69.3°$

$$\left(\frac{m}{m_c}\right)^2 = \frac{0.125}{0.2^2} + \frac{7}{8}\frac{1}{0.2} = 7.5, \Rightarrow \frac{m_c}{m} \simeq 0.365$$

$$\Rightarrow H = 8.6\times10^{3}\times0.365 = \cancel{1\times10^{3}\text{ Gauss}}$$
$$= 3.14\times10^{3} \text{ Gauss}$$

This is the upper electron peak in Fig. 28.9.

4. (a)



At $T=0$, $n_c = 0$, $p_v = 0$

N_a electrons drop from the donor levels to fill the acceptor levels

$\Longrightarrow$ $n_d = N_d - P_a$, $p_a = 0$

Note that $n_d - N_d = -p_a$ or $n_c + n_d - N_d = p_v + p_a - P_d$ ①

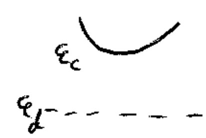
(b) Eq. ① above still holds at $T=0$ because each thermal excitation of an electron creates a hole (and so increases p_v or p_a by 1) and an electron (and so increases n_c or n_d by 1).

(c) At $T=0$ $n_c = p_v = 0$

$n_d = 0$, $p_a = P_a - N_d$.

(d) Same logic as for part (b)

5. (a) From Eq. (4) with $P_a = p_a = 0$ (since there are no acceptors) and $p_v = 0$ (since we are told that no electrons are excited from the valence band), we have.



$$n_c(T) = N_d - n_d(T).$$

(b) From the discussion in Qu. (1) we have.

$$n_d(T) = \frac{N_d}{1 + \frac{1}{2} e^{\beta(\varepsilon_d - \mu)}}$$

and, using standard bookwork

$$n_c(T) = e^{-\beta(\varepsilon_c - \mu)} N_c(T) \quad \text{where} \quad N_c(T) = \frac{1}{4}\left(\frac{2 m_c k_B T}{\pi \hbar^2}\right)^{3/2},$$

so $N_c(T)\,e^{-\beta(\varepsilon_c-\mu)} = N_d\left[1-\dfrac{1}{\frac{1}{2}e^{\beta(\varepsilon_d-\mu)}+1}\right]$

(c) This is a quadratic equation for $\lambda = e^{\beta\mu}$.

$$\frac{N_c e^{-\beta\varepsilon_c}}{N_d}\lambda = 1-\frac{1}{\frac{1}{2}e^{\beta\varepsilon_d}\lambda^{-1}+1}$$

$$\Rightarrow \frac{N_c}{N_d}e^{-\beta\varepsilon_c}\lambda = \frac{\frac{1}{2}e^{\beta\varepsilon_d}+\lambda-\lambda}{\frac{1}{2}e^{\beta\varepsilon_d}+\lambda} = \frac{1}{2}\frac{e^{\beta\varepsilon_d}}{\lambda+\frac{1}{2}e^{\beta\varepsilon_d}}$$

$$\lambda\left(\lambda+\tfrac{1}{2}e^{\beta\varepsilon_d}\right) = \tfrac{1}{2}x\,e^{2\beta\varepsilon_d} \quad \text{where } x = \frac{N_d}{N_c(T)}e^{\beta(\varepsilon_c-\varepsilon_d)}$$

$$\lambda = \frac{-\frac{1}{2}e^{\beta\varepsilon_d} \pm \sqrt{\frac{1}{4}e^{2\beta\varepsilon_d}+2x\,e^{2\beta\varepsilon_d}}}{2},$$

need the + sol^n since $\lambda > 0$

$$\lambda = e^{\beta\varepsilon_d}\left[\frac{-1+\sqrt{1+8x}}{4}\right]$$

$$\Rightarrow \mu(T) = \varepsilon_d + k_BT\ln\left[\frac{-1+\sqrt{1+8x(T)}}{4}\right]$$

So define T_d by $N_c(T) = N_d\left(\frac{T}{T_d}\right)^{3/2}$

So $T_d = \dfrac{\pi\hbar^2}{2m_ck_B}(4N_D)^{2/3}$

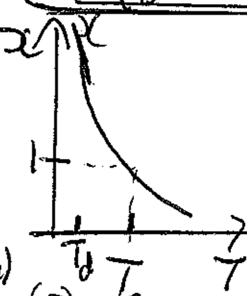
i.e. $N_c(T_d) = N_d$

and $x = \left(\frac{T_d}{T}\right)^{3/2}e^{\beta(\varepsilon_c-\varepsilon_d)}$

[Note: 2 energy scales, T_d set by the impurity density, and $\varepsilon_c-\varepsilon_d$ set by the impurity energy]

Now $\varepsilon_c > \varepsilon_d$ and $\dfrac{1}{N_c(T)} \propto T^{-3/2}$

so $x(T)$ is a monotonically decreasing f^n of T



(d) let $x(T_0)=1$, this defines T_0. Note that $T_0 > T_d$, $\left(\frac{T_0}{T_d}\right)^{3/2} = e^{\beta_0(\varepsilon_c-\varepsilon_d)}$ (0)

when $T \gg T_0$, i.e. $x(T) \ll 1$, $\ln\left[\dfrac{-1+\sqrt{1+8x}}{4}\right] = \ln\left[\dfrac{-1+1+4x\ldots}{4}\right]$

$= \ln x$. which is large and negative

Hence $\dfrac{\mu(T)-\varepsilon_d}{k_BT}$ $\underset{\simeq \ln x}{}$ is large and negative ~~or more precisely~~

$\boxed{\text{Note that } \mu = (\varepsilon_c \;\cancel{}) - \tfrac{3}{2}k_BT\ln(T/T_0)},$

so $e^{-\beta(\varepsilon_d-\mu(T))} \ll 1$ so $\dfrac{n_d(T)}{N_d} \ll 1$ and $n_c(T) = N_d$ from part (a) and $T \ll T_D$,

(e) For $T \ll T_0$, i.e. $x(T) \gg 1$, $\ln\left[\frac{-1+\sqrt{1+8x}}{4}\right] \simeq \frac{1}{2}\ln\left(\frac{x}{2}\right)$

$$\Rightarrow \mu(T) = \varepsilon_d + \frac{1}{2}(\varepsilon_c - \varepsilon_d) + \frac{k_B T}{2}\ln\left(\frac{N_d}{2N_c}\right)$$

$$= \left(\frac{\varepsilon_c + \varepsilon_d}{2}\right) + \frac{k_B T}{2}\ln\left[\frac{N_d}{2N_c(T)}\right] = \left(\frac{\varepsilon_c + \varepsilon_d}{2}\right) + k_B T\left[\frac{3}{4}\ln(T_D/T) - \frac{1}{2}\ln 2\right] \quad (1)$$

i.e. for $T \to 0$ μ is half way in between ε_c and ε_d

From Eq. (1) and part (b)

$$n_c(T) = N_c(T)\, e^{-\beta(\varepsilon_c - \mu)} = \sqrt{\frac{N_c(T) N_d}{2}}\; e^{-\beta(\varepsilon_c - \varepsilon_d)/2} = \frac{N_d}{\sqrt{2}}\left(\frac{T}{T_d}\right)^{3/4} e^{-\beta\frac{(\varepsilon_c - \varepsilon_d)}{2}}$$

(f) (i) $T = 300K$ $\quad \varepsilon_c - \varepsilon_d = 2\,\text{meV} \simeq 23K$.

$$N_c(T) = 2.5\left(\frac{m_c}{m}\right)^{3/2}\left(\frac{T}{300}\right)^{3/2} \times 10^{19}/\text{cm}^3 \quad \text{from AM Eq. (28.16)}$$

$$= 2.5\left(\frac{T}{300}\right)^{3/2} \times 10^{16}/\text{cm}^3$$

$$= 2.5 \times 10^{16}$$

$$x = \frac{N_d}{N_c(T)}\, e^{\beta(\varepsilon_c - \varepsilon_d)} = \frac{1}{25}\, e^{\frac{23}{300}} \ll 1,$$ i.e. the high-T regime.

$$\Rightarrow n_c(T) \simeq N_d = 10^{15}/\text{cm}^3$$

(ii) $T = 4K$

$$N_c(T) = 2.5\left(\frac{4}{300}\right)^{3/2} \times 10^{16} = 3.85 \times 10^{13}$$

$$\text{so } x = \frac{10^{15}}{3.85 \times 10^{13}} \exp\left[\frac{23}{4}\right] \gg 1,$$ i.e. the low-T regime

$$n_c(T) \simeq \sqrt{\frac{N_c(T) N_d}{2}}\; e^{-\beta(\varepsilon_c - \varepsilon_d)/2} \simeq 7.6 \times 10^{12}/\text{cm}^3$$

i.e. much less than in part (i)

Note: For comparison T_d is given by $N_c(T_d) = N_d$

i.e. $2.5 \times \left(\frac{m_c}{m}\right)^{3/2}\left(\frac{T_d}{300}\right)^{3/2} \times 10^{19} = 10^{15}$, $\Rightarrow (T_d/300)^{3/2} = \frac{1}{25}$

$\Rightarrow T_d = 35K$. Solving Eq. (0) for T_0 gives $\Rightarrow T_0 = 48K$.