(25)

Physics 231 Homework 5 Solutions

1/ $$\mathcal{E}(\vec{k}) = \mathcal{E}(0) + \frac{\hbar^2}{2}\left(\frac{k_x^2}{m_x} + \frac{k_y^2}{m_y} + \frac{k_z^2}{m_z}\right)$$

Assume $\vec{E} = 0$, and $\vec{H}$ is in the z-direction

$$\vec{v} = \frac{1}{\hbar}\vec{\nabla}_k \mathcal{E}_k = \hbar\left(\frac{k_x}{m_x}, \frac{k_y}{m_y}, \frac{k_z}{m_z}\right) \quad (1)$$

$$m_x \frac{dv_x}{dt} = (-e)\frac{v_y}{c}H \Rightarrow \hbar\frac{dk_x}{dt} = -e\frac{\hbar k_y}{c m_y}H \quad (2)$$

$$m_y \frac{dv_y}{dt} = (-e)\left(-\frac{v_x}{c}\right)H \Rightarrow \hbar\frac{dk_y}{dt} = e\frac{\hbar k_x}{c m_x}H \quad (3)$$

Differentiate (2) w.r.t. t.

$$\frac{d^2k_x}{dt^2} = -\frac{e}{c}\frac{H}{m_y}\frac{d}{dt}k_y = -\frac{(eH)^2}{c^2 m_x m_y}k_x.$$

$\Rightarrow$ periodic motion

(3) i.e. $\omega = \frac{eH}{\sqrt{m_x m_y}\,c} = \frac{eH}{m^* c}$ where $m^* = (m_x m_y)^{1/2}$

2/ $$T(\mathcal{E}, k_z) = \frac{\hbar^2 c}{eH}\frac{\partial}{\partial\mathcal{E}}A(\mathcal{E}, k_z)$$ For a derivation see AM p.231-233

For free electrons $\mathcal{E} = \frac{\hbar^2}{2m}(k_z^2 + k_\perp^2)$, where $k_\perp^2 = k_x^2 + k_y^2$.

Area $A = \pi k_\perp^2 = \pi\left(\frac{2m}{\hbar^2}\mathcal{E} - k_z^2\right)$

Hence $\frac{\partial}{\partial\mathcal{E}}A(\mathcal{E}, k_z) = \frac{2m\pi}{\hbar^2}$.

(3) $$\Rightarrow T = \frac{\hbar^2 c}{eH}\frac{2m\pi}{\hbar^2} = 2\pi\frac{mc}{eH}$$

$$= \frac{2\pi}{\omega_c} \text{ where } \omega_c = \frac{eH}{mc}, \text{ the cyclotron frequency}$$

3. $A\left(\mathcal{E}_\nu(k_z), k_z\right) = \left(\nu+\frac{1}{2}\right)\frac{2\pi e}{\hbar c} H$

(a) As in Qu. 2. $A = \pi\left[\frac{2m}{\hbar^2}\mathcal{E} - k_z^2\right]$

$$\text{so } \pi\left[\frac{2m\mathcal{E}}{\hbar^2} - k_z^2\right] = \left(\nu+\frac{1}{2}\right)\frac{2\pi e}{\hbar c} H.$$

$$\mathcal{E} = \frac{\hbar^2 k_z^2}{2m} + \left(\nu+\frac{1}{2}\right)\frac{2\pi e}{\hbar c}\frac{1}{\pi}\frac{\hbar^2}{2m} H$$

$$= \frac{\hbar^2 k_z^2}{2m} + \left(\nu+\frac{1}{2}\right)\hbar\underbrace{\frac{eH}{mc}}_{\omega_c}$$

3 (b) For area of k-space ΔA, the number of levels is

$$\xrightarrow{\text{spin}} 2\frac{A}{(2\pi)^2}\Delta A = 2\frac{A}{4\pi^2}\frac{2\pi e}{\hbar c}H = \frac{2e}{hc}HA$$

(counting states in a box)

4. Count the number of rapid oscillations between 2 successive minima of the envelope.

I estimate 52

3 $\Rightarrow \frac{A_{belly}}{A_{neck}} \simeq 52$, see AM Fig. 15-6.

5. Probability of a collision in time dt is $\frac{dt}{\tau}$

(a) Let $Q(t)$ be the probability that it had not collision during the preceding t secs

Then $\underbrace{Q(t+dt)}_{\text{Prob. no coll. up to } t+dt} = \underbrace{\left(1-\frac{dt}{\tau}\right)}_{\text{Prob. no coll. between } t \text{ and } t+dt}\underbrace{Q(t)}_{\text{prob. no coll. up to time } t}$

Hence $\dfrac{Q(t+dt) - Q(t)}{dt} = -\dfrac{Q(t)}{\tau}$

or $\dfrac{dQ}{dt} = -\dfrac{Q}{\tau} \Rightarrow \underline{Q(t) = e^{-t/\tau}}$ since $Q(0) = 1$.

For collisions ~~between~~ during the next t seconds the argument is the same.

n.b. Let the probability that the next collision is between t and $t+dt$ be $P(t)dt$. Then

$$P(t)\,dt = \underbrace{Q(t)}_{\text{prob. of no collision up to time } t}\ \underbrace{\frac{dt}{\tau}}_{\text{prob. of a collision between } t \text{ and } t+dt}$$

Hence $P(t) = \dfrac{Q(t)}{\tau} = \underline{\dfrac{e^{-t/\tau}}{\tau}}$

Similarly for probability of the time back to the last collision.

(b) Let one electron have a collision at time $t=0$. Let the probability that it did not have another collision up to time t be $Q(t)$.
Repeating the argument of part (a) $Q(t) = e^{-t/\tau}$
Similarly the probability that the next collision is between t and $t+dt$ is $P(t)\,dt$ where

$$\underline{P(t) = \frac{Q(t)}{\tau} = \frac{e^{-t/\tau}}{\tau}}$$

(c) Average over the distribution $P(t)$ from part (a)
Hence mean time back to last collision = mean time to next collision (averaged over electrons) $= \int_0^\infty t\,\dfrac{e^{-t/\tau}}{\tau}\,dt$

$$\underline{\underline{= \tau}}$$

(d) ~~Similarly, as in (c),~~ Using part (b) the mean time between successive collisions of an electron is

$$\int_0^\infty t\,\frac{e^{-t/\tau}}{\tau}\,dt = \underline{\underline{\tau}}$$

(e) There seems to be a difference of a factor of 2 between the results of (c) and (d) since

- (c) $\rightarrow$ average time between last and next collision (averaged over all electrons at a single moment of time) $= \tau + \tau = \underline{\underline{2\tau}}$

- (d) $\rightarrow$ average time between collisions (averaged over collisions for a single electron) $= \underline{\tau}$

Note, however that the way the averages are done
5 (indicated by the last 2 lines above in brackets) are
<u>different</u>.

In (d) we give equal weight to all intervals between collisions.

In (c) we give <u>more</u> weight to the long intervals because at a fixed time we are more likely than on average to catch an electron in one of its long intervals than one of its short intervals (simply because it lasts longer).

To check this let us calculate the probability distribution $\tilde{P}(t)$ that the time between last and next collision, averaged over electrons (at a fixed time), is t. Let t_1 be the time back to the last collision and t_2 be the time to the next collision, so

$$t = t_1 + t_2$$

and $\tilde{p}(t) = \int_0^\infty p(t_1)\, p(t_2)\, \delta(t - t_1 - t_2)\, dt_1\, dt_2.$

Do the integral over t_2 say

$$\tilde{p}(t) = \int_0^t p(t_1) P(t - t_1) dt_1 = \int_0^t \frac{e^{-t_1/\tau}}{\tau} \frac{e^{-(t-t_1)/\tau}}{\tau} dt_1$$

$$= \frac{e^{-t/\tau}}{\tau^2} \int_0^t dt_1 = \underline{\frac{1}{\tau^2} t e^{-t/\tau}}$$

Notice the extra factor of (t/τ) which comes from weighting the long time intervals more than the short time intervals.

Take the average of $\tilde{p}(t)$

$$\int_0^\infty t \tilde{p}(t) dt = \frac{1}{\tau^2} \int_0^\infty t^2 e^{-t/\tau} dt = \underline{\underline{2\tau}} \quad \text{(Standard integral)}$$

i.e. we recover the extra factor of 2.

6. Surface plasmon

Look for a solution to maxwell's equations of the form

$$E_x = A e^{iqx} e^{-kz}, \quad E_y = 0, \quad E_z = B e^{iqx} e^{-kz} \quad (z > 0)$$

$$E_x = C e^{iqx} e^{k'z}, \quad E_y = 0, \quad E_z = D e^{iqx} e^{k'z} \quad (z < 0)$$

where $z > 0$ is metal $\Rightarrow \left\{ \begin{array}{l} \epsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega} \\ \text{where } \sigma(\omega) = \frac{ne^2\tau}{m} \frac{1}{1 - i\omega\tau} \end{array} \right.$

$z < 0$ is vacuum

(a)

Since we require a solution to Maxwell's equations we must have.

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\epsilon \vec{E}) = 0$$

In each medium, ε is independent of $\vec{r}$ so (6)

$$\vec{\nabla}\cdot\vec{E}=0$$

$$\Rightarrow \quad iqA - kB = 0 \qquad \text{(from } z>0\text{)} \quad (1)$$

$$iqC + k'D = 0 \qquad \text{(from } z<0\text{)} \quad (2)$$

Also we must satisfy the boundary conditions.

$E_{\parallel}$ continuous so $A = C$ (3)

$(\varepsilon E)_{\perp}$ continuous so $\varepsilon(\omega)B = D$ (4)

From (2) – (4) $iqA + k'\varepsilon(\omega)B = 0$

Compare with (1) $\Longrightarrow$ $\underline{\underline{k = -k'\varepsilon(\omega)}}$ (5)

We also require that

$$-\nabla^2\vec{E} = \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}$$

since the solution must satisfy Maxwell's equations.

For $z<0$ this gives (6)

$$\underline{\underline{q^2 - k'^2 = \frac{\omega^2}{c^2}}}$$

For $z>0$ this gives (7)

$$\underline{\underline{q^2 - k^2 = \varepsilon(\omega)\frac{\omega^2}{c^2}}}$$

The 3 equations are $\underline{\underline{(5), (6) \text{ and } (7)}}$

(b) From (6) $k'^2 = q^2 - \frac{\omega^2}{c^2}$ (8)

From (5) and (7)

$$(k')^2 = \frac{q^2}{\varepsilon(\omega)^2} - \frac{\omega^2}{c^2\varepsilon(\omega)} \quad (9)$$

From (8) and (9) $q^2 - \frac{\omega^2}{c^2} = \frac{q^2}{\varepsilon(\omega)^2} - \frac{\omega^2}{c^2\varepsilon(\omega)}$

so $\quad q^2(1-\varepsilon^2) = \frac{\omega^2}{c^2}\,\varepsilon(\varepsilon-1)$

or $\quad q^2 = -\frac{\omega^2}{c^2}\,\frac{\varepsilon}{\varepsilon+1} = -\frac{\omega^2}{c^2}\,\frac{1}{1+1/\varepsilon(\omega)}$

or $\quad \omega^2 = c^2 q^2\left[1+\frac{1}{\varepsilon(\omega)}\right]$ $\quad$ 10

(c) $\quad \varepsilon(\omega) = 1 + \frac{4\pi i\sigma(\omega)}{\omega}\,, \qquad \sigma(\omega) = \frac{ne^2\tau}{m}\,\frac{1}{1-i\omega\tau}$

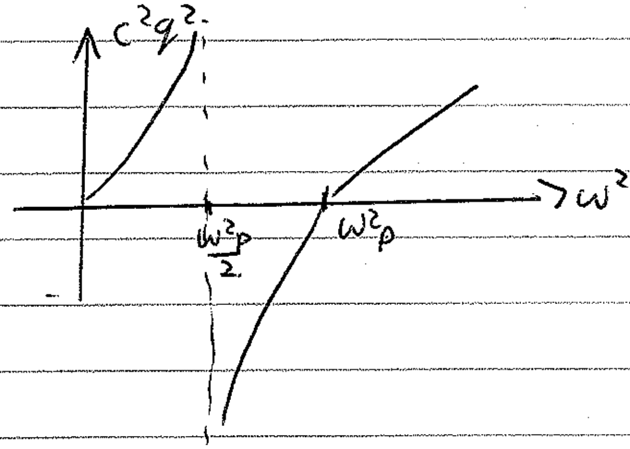
For $\omega\tau \gg 1$, $\quad \sigma(\omega) = \frac{i\,ne^2}{m\omega}$

so $\quad \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$ $\quad$ 11 $\quad$ where $\omega_p^2 = \frac{4\pi ne^2}{m}$

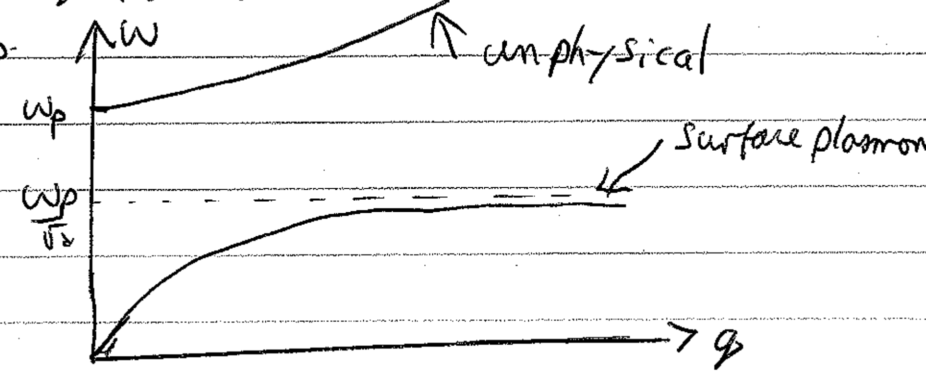
ω_p is the plasma frequency

(d) $\quad c^2q^2 = \frac{\omega^2}{1+\frac{1}{1-\omega_p^2/\omega^2}} = \frac{\omega^2(1-\omega_p^2/\omega^2)}{1+1-\omega_p^2/\omega^2}$

$= \omega^2\left(\frac{\omega^2-\omega_p^2}{2\omega^2-\omega_p^2}\right)$



(e) n.b. require $c^2q^2 > 0$ so there is no real solution for $\frac{\omega_p}{\sqrt{2}} < \omega < \omega_p$



(f) $\frac{c^2q^2}{\omega^2} = \frac{\omega^2-\omega_p^2}{2\omega^2-\omega_p^2}$. For $\frac{cq}{\omega} \gg 1$, $\omega = \frac{\omega_p}{\sqrt{2}}$.

When $\omega = \frac{\omega_p}{\sqrt{2}}$, $\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} = 1-2 = -1$ ← from (11)

From (11) $\frac{\omega^2}{c^2} - q^2 = \frac{q^2}{\varepsilon(\omega)} = -q^2$

But from (6) $\frac{\omega^2}{c^2} - q^2 = -h'^2$ so $h' = |q|$.

From (5) $h = h' \; (= |q|)$.

Hence h, h', q are all real and h, h' are positive.

Hence this is truly a surface wave. The amplitude decays exponentially both for $z > 0$ and $z < 0$.

Propogation in x direction
$\vec{E}$ has components in x and z directions so it is mixed transverse and longitudinal).
$\frac{B}{A} = \frac{iq}{h} = i$ so the wave is, in a sense, circularly polorized.

(g) If $\omega > \omega_p$, $\varepsilon(\omega) > 0$ so, from (5)
both h and h' can not be positive.
(8) Hence amplitude ~~blows up both at z~~ cannot go to zero both for $z \to \infty$ and $z \to -\infty$, and so the solution is unphysical.