Total (31)

Physics 231 Homework 3 Solutions

1/ Intensity of Bragg scattering is proportional to

(a)/ e^{-2W}

where $W = \frac{1}{N} \sum_{\vec{k}, s} \frac{\hbar}{2M\omega_{s,\vec{k}}} \left(\vec{G} \cdot \vec{\varepsilon}_{s,\vec{k}}\right)^2 \left[n(\omega_{s,\vec{k}}) + \frac{1}{2}\right]$

averaging over directions, which gives an unimportant numerical factor of order unity, and replacing the sum by an integral gives

$$W \propto \int_0 dk\, k^{d-1} \frac{1}{\omega_k} \left[\frac{1}{e^{\beta\hbar\omega_k} - 1} + \frac{1}{2} \right]$$

Now $\omega_k \propto k$ and we are interested in the potent divergent behavior of the integrand as $k \to 0$, so

~~10/9~~ (8)

$$\underline{W \propto \int_0 \frac{dk}{k} k^{d-1} \left[\frac{T}{k} + \frac{1}{2} \right]}$$ neglecting constants

For $\underline{T > 0}$ the T/k term dominates over $1/2$

$$W \propto T \int_0 \frac{dk}{k} k^{d-2}$$

$\underline{\text{In } d = 2}$ $\quad W \propto T \int_0 \frac{dk}{k}$ which $\underline{\text{diverges logarithmically}}$

$\underline{\text{In } d = 1}$ $\quad W \propto T \int_0 \frac{dk}{k^2}$ " $\underline{\text{diverges linearly}}$

n.b. in $d = 3$ $\quad W$ is finite.

(b)/ $\underline{T = 0}$ $\quad W \propto \int_0 \frac{dk}{k} k^{d-1}$

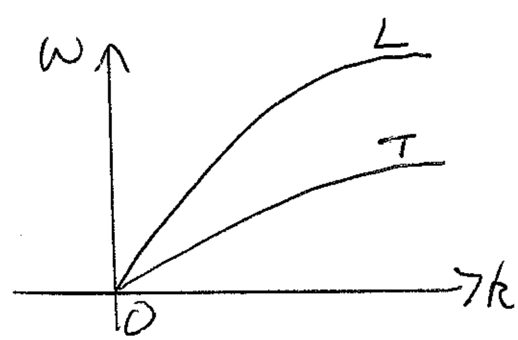
$\underline{\text{In } d = 2}$ $\quad W \propto \int_0 dk \,\text{const.}$ which is $\underline{\text{finite}}$.

$\underline{\text{In } d = 1}$ $\quad W \propto \int_0 \frac{dk}{k}$ which diverges $\underline{\text{logarithmically}}$.

(c) If $W = \infty$, $e^{-2W} = 0$ so there is no translational long range order when W diverges (though one has "quasi long range order" in 2-d (at finite-T), see the hand out.

2/. Dispersion relation

$$\frac{d^2\omega}{dk^2} < 0$$



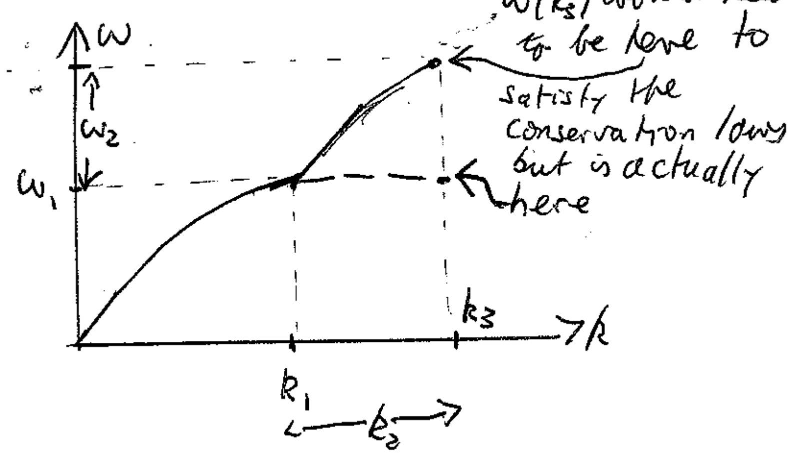
3 phonon processes

$$\omega_1 + \omega_2 = \omega_3$$

$$k_1 + k_2 = k_3 \ (+G)$$

(a) all phonons belong to the same branch.

Graphical construction ⟹



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(b) Since $\omega_L(k) > \omega_T(k)$ the process

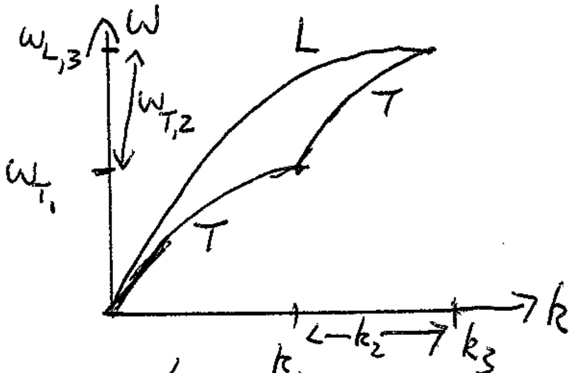
$$L + L \Leftrightarrow T$$

also does not work, because the mismatch in frequencies (if k conservation is satisfied) is even greater.

Similarly for $L + T \Leftrightarrow T$.

In order to make up the mismatch in frequencies the single phonon must be in a branch higher than at least one of the members of the pair, eg

$T + T \Leftrightarrow L \quad \Longrightarrow$



Similarly $T + L \Leftrightarrow L$ works

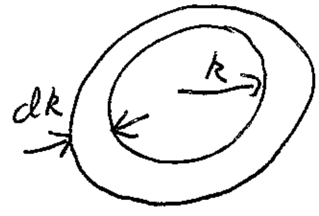
3 (a) Density of points in k-space / unit volume is $\frac{1}{(2\pi)^2}$

Hence $n = \underset{\text{spin}}{2} \; \underbrace{\pi k_F^2}_{\text{area of Fermi circle}} \; \frac{1}{(2\pi)^2} = \frac{k_F^2}{2\pi}$

(b) $n = \frac{1}{\pi r_s^2}$ $\qquad$ πr_s^2 is the area per particle.

i.e $\frac{1}{\pi r_s^2} = \frac{k_F^2}{2\pi}$ $\qquad$ or $\quad k_F = \frac{\sqrt{2}}{r_s}$

(c) $g(\varepsilon)\,d\varepsilon = \underset{\text{spin}}{2} \; \underbrace{2\pi k \, dk}_{\text{area of shell}} \; \frac{1}{(2\pi)^2}$



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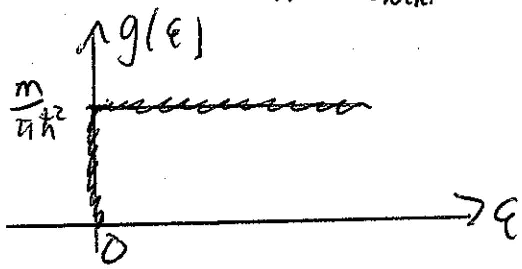
$g(\varepsilon) = \frac{k}{\pi} \frac{1}{d\varepsilon/dk} = \frac{k\, m}{\pi \hbar^2 k} = \frac{m}{\pi \hbar^2}$, a constant (for $\varepsilon > 0$)



(d) Sommerfeld expansion

$$n = \int_0^\infty f(\varepsilon) g(\varepsilon)\, d\varepsilon = \int_0^\mu g(\varepsilon)\, d\varepsilon + \sum_{n=1}^{\infty} a_n (k_B T)^{2n} \frac{d^{2n-1}}{d\varepsilon^{2n-1}} g(\varepsilon)\Big|_\mu$$

But all derivatives vanish so

$n = \int_0^\mu g(\varepsilon)\, d\varepsilon$ $\qquad$ according to the Sommerfeld expansion

At $T=0$, $n' = \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon$ $\qquad$ so $\mu(T) = \varepsilon_F$ according to Somm. expansion.

(e) Does this mean that $\mu(T)$ is really identical to $\varepsilon_F \; (\equiv \mu(T=0))$?

NO. It just means that all terms that are <u>powers</u> of T vanish. As we shall see in this ~~next~~ and the next section there are corrections which vanish <u>exponentially</u> as $T \to 0$.

Now $\quad n = \dfrac{m}{\pi\hbar^2}\displaystyle\int_0^{\varepsilon_F} d\varepsilon = \dfrac{m\,\varepsilon_F}{\pi\hbar^2}$ (1) $(T=0)$

and $\quad n = \dfrac{m}{\pi\hbar^2}\displaystyle\int_0^{\infty} \dfrac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon = \dfrac{m k_B T}{\pi\hbar^2}\displaystyle\int_{-\beta\mu}^{\infty}\dfrac{dx}{e^{x}+1}$ $\quad (T>0)$

Now $I = \displaystyle\int \dfrac{dx}{e^x+1} = \int\left[1 - \dfrac{e^x}{e^x+1}\right]dx = x - \ln(1+e^x)$ so

$$n = \frac{m k_B T}{\pi\hbar^2}\Bigg[\underbrace{\frac{\mu}{k_B T} + \ln\left(1+e^{-\mu/k_B T}\right)}_{\ln[1+e^{\beta\mu}]}\Bigg] \quad (2)$$

, so from (1) and (2) $\; \underline{e^{\beta\varepsilon_F} = e^{\beta\mu}+1}$

Equate (1) and (2)

$$\underline{\underline{\varepsilon_F = \mu + k_B T \ln\left[1 + e^{-\mu/k_B T}\right]}} \quad (3')$$

can also be written $\; \underline{\underline{\mu = k_B T \ln\left[e^{\beta\varepsilon_F} - 1\right]}}$ (3)

(f) As $T \to 0$, μ is ~~very~~ very close to ε_F.
Also $e^{-\mu/k_B T}$ is very small and so, to leading order in small quantities, we can replace it by $e^{-\varepsilon_F/k_B T}$

Hence $\mu(T) \simeq \varepsilon_F - k_B T \ln\left[1 + e^{-\varepsilon_F/k_B T}\right]$

$$\underline{\underline{\simeq \varepsilon_F - k_B T\, e^{-\varepsilon_F/k_B T} \ldots}}$$

i.e. $\mu - \varepsilon_F$ is <u>exponentially</u> small as $T \to 0$

(g) (Not asked this year) at high T, from (3) $\quad \mu = k_B T \ln(\beta\varepsilon_F) = k_B T \ln\left(\dfrac{n\pi\hbar^2}{m k_B T}\right) \quad (<0)$

$= \underline{k_B T \ln(n A_Q)}$ $\qquad A_Q = \dfrac{\pi\hbar^2}{m k_B T}$ is the square of the thermal de Broglie wavelength

4. $e^{\mu/k_BT} \ll 1$ for classical statistics

Use the classical expression

(a) $n = \int_0^\infty e^{-(\varepsilon-\mu)/k_BT} g(\varepsilon)\, d\varepsilon.$

Now $g(\varepsilon) = \frac{\sqrt{2}}{\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$ so

$$n = \frac{\sqrt{2}}{\pi^2}\left(\frac{m}{\hbar^2}\right)^{3/2} e^{\beta\mu} \underbrace{\int_0^\infty \varepsilon^{1/2} e^{-\beta\varepsilon}\, d\varepsilon}_{(k_BT)^{3/2} \underbrace{\int_0^\infty x^{1/2} e^{-x}\, dx}_{\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}}}$$

i.e. $n = \frac{1}{\sqrt{2}}\left(\frac{m\pi k_BT}{\hbar^2}\right)^{3/2} e^{\mu/k_BT}$

But this is only valid if $e^{\mu/k_BT} \ll 1$

i.e. $n \ll \frac{1}{\sqrt{2}}\left(\frac{m\pi k_BT}{\hbar^2}\right)^{3/2}$

(b) But $n = \frac{1}{\frac{4\pi}{3} r_s^3}$, r_s is the typical inter particle spacing

i.e. $r_s \gg \left(\frac{\hbar^2}{2mk_BT}\right)^{1/2}$ (1) (dropping factors of order unity)

(b) $\left(\frac{\hbar^2}{2mk_BT}\right)^{1/2}$ is ~~of~~ the thermal de-Broglie wavelength,

i.e. the de Broglie wavelength of a particle whose energy is k_BT (times $\frac{1}{2\pi}$) $\left(k_{deB} = \frac{1}{\lambda_{deB}}\right)$

i.e. $\frac{\hbar^2}{2m}\left(\frac{1}{\lambda_{deB}}\right)^2 = k_BT$

(c) ~~just plug in the numbers~~ See next page

5 $\quad V_F = \frac{\hbar}{m} k_F.$

(a) Roughly, $k_F \sim \frac{1}{a_0}$ where a_0 is the Bohr radius (i.e. of order the interatomic separation)

$$a_0 = \frac{\hbar^2}{me^2}.$$

Hence $V_F \simeq \frac{\hbar}{m}\frac{me^2}{\hbar^2} = \frac{e^2}{\hbar} = \frac{e^2}{\hbar c} c = \underline{\underline{\alpha c}}$

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where $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}$ is the fine structure constant

(b) The ions are roughly 4–5 orders of magnitude heavier than the electrons and, in general, $\omega \propto M^{-1/2}$ (see e.g. the simple harmonic oscillator)

4 (c) $\quad \hbar = 1.05\times10^{-34}, \quad m = 9.1\times10^{-31}, \quad k_B = 1.38\times10^{-23}$

$$a_0 = 0.53\times10^{-10}$$

Hence condition ① on p.5 is

$$\frac{r_s}{a_0} \gg \left(\frac{\hbar^2}{2mk_BTa_0^2}\right)^{1/2} = \left[\frac{\left(1.05\times10^{-34}\right)^2}{2\times9.1\times10^{-31}\times1.38\times10^{-23}\times(0.53\times10^{-10})^2T}\right]^{1/2}$$

$$= \left[\frac{1.6\times10^5}{T}\right]^{1/2} \simeq \underline{\underline{\left[\frac{10^5}{T}\right]^{1/2}}}$$