PHYSICS 219

Homework 3

Due in class, Wednesday May 10

1. *Transition temperature for Bose-Einstein condensation*
   Consider a system of non-interacting spinless bosons. Determine the temperature at which Bose-Einstein condensation occurs, expressing your answer in terms of the density of particles, n, their mass, m, and $\hbar$.
   Note: You need to show your working, including evaluation of the integral.

2. *Exact solution of the hard-core gas in one-dimension.*
   Consider a "hard-sphere" model of a classical gas in which the particles have radius $a/2$ so the potential, $U(r)$, is given by
   $$U(r) = 0\ , \quad r > a,$$
   $$U(r) = \infty\ , \quad r < a.$$
   Note that the particles cannot come within $a/2$ of the walls.
   In one dimension it is fairly straightforward to compute the equation of state exactly. Show that the result in this case is just the Van der Waals equation for a hard sphere gas,
   $$P(V - Nb) = Nk_BT,$$
   or, equivalently,
   $$\frac{P}{k_BT} = \frac{n}{1 - bn}, \tag{1}$$
   where $b = a$. (*Note:* Nb is just the total "volume" of the particles.) To obtain this result, compare expressions for the partition function for N particles confined in one-dimension (a) when they are non-interacting and (b) when they have the above potential.
   n.b. A special feature of one-dimension is that one can *order* the particles.

3. *Virial expansion of the hard-core gas in one and three dimensions.*
   In more that one dimension the equation of state of the hard-core gas described in the previous problem can not be evaluated exactly. Hence compute the virial coefficients, B_2 and B_3 both for one and three dimensions. Check that your results for one dimension agree with the answer to the previous question. Also form the dimensionless ratio B_3/B_2^2 in both cases. Note that, since $B_3/B_2^2 \neq 1$ in three dimensions, the equation of state *cannot* be expressed in the form Eq. (1) above for $d = 3$, even though one might naively have expected this.

4. *Virial coefficients of non-interacting quantum particles.*
   Determine the virial coefficient, B_2 for the case of a *non-interacting* gas of (a) fermions, and (b) bosons. Your answer should involve the spin degeneracy factor $2S + 1$. Discuss the sign of B_2 in the two cases.
   Note:
   (a) The virial coefficient B_2 is defined by $P/k_BT = n + B_2n^2 + \dots$, where $n \equiv N/V$ is the particle density.
   (b) For the gas of non-interacting quantum particles the virial expansion describes the deviation from *classical* behavior.

5. *Van der Waals equation of state.*
Consider the van der Waals equation of state for a fluid:

$$\left(P + a\left(\frac{N}{V}\right)^2\right)(V - Nb) = Nk_BT \ ,$$

where the parameters a and b depend on the fluid in question.

(a) Sketch the isotherms in a P–V plot.

(b) Determine the values of the critical pressure, P_c, critical volume, V_c, and critical temperature, T_c.

(c) Show that the van der Waals equation of state can be expressed entirely in terms of the reduced dimensionless variables,

$$v = \frac{V}{V_c} \ , \quad p = \frac{P}{P_c} \ , \quad t = \frac{T}{T_c} \ ,$$

without making any explicit reference to the parameters a and b. This is called the *law of corresponding states.*

(d) Show that for $t < 1$, the region where liquid and gas phases coexist, one naively finds that $\partial p/\partial v > 0$, which is unphysical because it corresponds to mechanical instability. Explain how this problem is rectified by the "Maxwell equal area construction".