PHYSICS 116C
Homework 6 Solutions

1. (a) Laplace's equation in circular polars is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0\,.$$

Writing the solution as

$$u(r,\theta)=\ R(r)\Theta(\theta)\,,$$

following standard steps, and multiplying by r^2, we get

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{d^2\Theta}{d\theta^2}=0\,.$$

Each piece must be a constant, which we write as $-n^2$, and so the equation for Θ is

$$\boxed{\frac{d^2\Theta}{d\theta^2}+n^2\Theta=0\,.}$$

The solution is

$$\boxed{\Theta=A_n\cos n\theta+B_n\sin n\theta}$$

with A_n and B_n arbitrary constants. n has to be an integer because we must get the $\boxed{\text{same solution when } \theta \text{ increases by } 2\pi}$ (since we get back to the same point in space.)

(b) The equation for R is

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)=n^2R$$

In this equation, each time we differentiate with respect to r we also multiply by r. As a result if we assume a solution involving a single power of r, e.g. $R=r^\lambda$, then each term will have this power. Substituting this guess into the equation gives

$$(\lambda^2-n^2)r^\lambda=0\,,$$

which is satisfied provided

$$\lambda=\pm n\,.$$

For $n\neq 0$, this gives us the two independent solutions we require. For $n=0$ it only gives one solution. We shall now see that the second solution involves a log. We start from the radial equation for $n=0$ (dividing by r)

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right)=0\,.$$

We can integrate it once to get

$$r\frac{dR}{dr}=C$$

or equivalently

$$\frac{dR}{dr}=\frac{C}{r}$$

where C is an arbitrary constant. Integrating this last equation then gives

$$R(r) = C \ln r + D$$

where D is another arbitrary constant.

Putting everything together, the general solution is

$$\boxed{u(r,\theta) = (C_0 + D_0 \ln r) + \sum_{n=1}^{\infty} \left(C_n r^n + \frac{D_n}{r^n} \right) (A_n \cos n\theta + B_n \sin n\theta)\,.}$$

Note that the isotropic solution (i.e. the solution independent of θ) is $\ln r$ (ignoring the trivial constant solution). Indeed, if we have an isotropic problem which satisfies Laplace's equation in two-dimensions, e.g. the electrostatic potential from an infinitely long line charge, the solution does vary as $\ln r$, see any textbook on E+M.

2. As discussed in class, the solution of Laplace's equation in a semi-infinite cylinder of radius a which satisfies $u = 0$ at $r = a$, is

$$u(r,\theta,z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n\left(c_{mn} \frac{r}{a}\right) (A_{mn} \cos n\theta + B_{mn} \sin n\theta)\, e^{-c_{mn} z/a}\,, \tag{1}$$

c_{mn} is the m-th zero of $J_n(x)$.

Note: Here I have not written out the derivation of Eq. (1) that was covered in class. However, if this question were asked in an exam, you would be expected to include the derivation, using the method of separation of variables, unless the question specifcially asked you not to.

The A_{mn} and B_{mn} are determined by the boundary condition at $z = 0$. Here this is $u = r \sin\theta$. By inspection we see that the A_{mn} are zero, as are the B_{nm} for $n \neq 1$. Hence, putting $z = 0$, and writing B_m for B_{m1} and c_m for c_{m1}, we have

$$r = \sum_{m=1}^{\infty} B_m J_1\left(c_m \frac{r}{a}\right)\,. \tag{2}$$

The coefficient B_n in this Bessel series is determined by multiplying by $rJ_1(c_n r/a)$ and using the orthogonality relation given in the question. This gives

$$B_n \int_0^a r J_1^2\left(c_n \frac{r}{a}\right) dr = \int_0^a r\, r\, J_1\left(c_n \frac{r}{a}\right) dx\,.$$

Setting $x = r/a$ this gives

$$B_n \int_0^1 x J_1^2(c_n x)\, dx = a \int_0^1 x\, x\, J_1(c_n x)\, dx\,. \tag{3}$$

The integral on the left hand side of Eq. (3) is given in the question to be $(1/2)J_2^2(c_n)$. To do the integral on the right of Eq. (3), we use the result given in the question that

$$\frac{d}{dx}\left[x^2 J_2(x)\right] = x^2 J_1(x)\,,$$

which, replacing x by $c_m x$, gives

$$\frac{1}{c_m}\frac{d}{dx}\left[x^2 J_2(c_m x)\right] = x^2 J_1(c_m x)\,.$$

Integrating this expression gives

$$\int_0^1 x^2 J_1(c_m x)\,dx = \frac{1}{c_m} J_2(c_m)\,.$$

Putting this into Eq. (3) gives

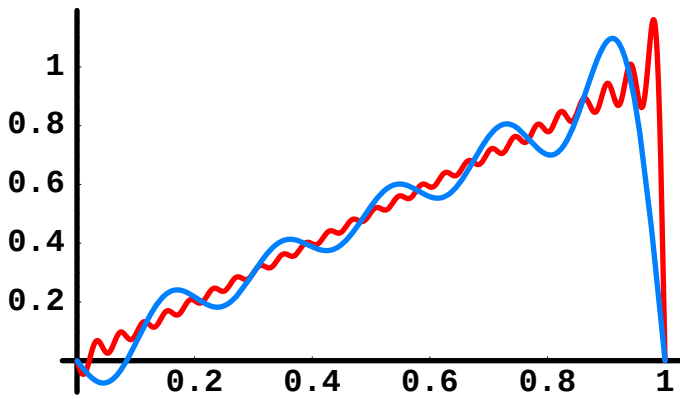
$$B_m = \frac{2a}{c_m J_2(c_m)}\,. \tag{4}$$

Hence the solution is

$$\boxed{u(r,\theta,z) = \sum_{m=1}^{\infty} \frac{2a}{c_m J_2(c_m)}\, J_1\left(c_m \frac{r}{a}\right) \sin\theta\, e^{-c_m z/a}\,,}$$

where, we recall, c_m is a zero of $J_1(x)$.

The figure below, in which the x-axis is r/a and the vertical axis is a^{-1} times the Bessel series for r in Eq. (2), uses the values for the B_m given in Eq. (4), including 10 terms (blue) and 50 terms (red). Note the oscillations (more rapid for 50 terms) and the rather slow convergence. This is due to the discontinuity at $x = a$. There is a similar phenomenon in Fourier series of functions with a discontinuity, see the discussion of the "Gibbs phenomenon" in Sec. 3.6 of Boas.



3. (a) Laplace's equation in cylindrical polar coordinates is

$$\frac{\partial^2 T}{\partial^2 r} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{5}$$

Separate out the z dependence by writing $T(r,\theta,z) = u(r,\theta)Z(z)$ which gives

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\frac{1}{u}\nabla_2^2 u\,, \tag{6}$$

where ∇_2^2 is the (two-dimensional) Laplacian in polar coordinates corresponding to the first three terms in Eq. (5). Each side of Eq. (6) must be a constant. Furthermore this separation

constant must be negative (which we call $-k^2$) in order to satisfy the boundary conditions that $Z(0) = Z(L) = 0$. This gives

$$\frac{d^2 Z}{dz^2} + k^2 Z = 0,$$

in which we need $k = \pi n/L$, so the solution is

$$Z(z) = \sin\left(\frac{n\pi z}{L}\right).$$

The equation for u is then

$$\boxed{\left(\nabla_2^2 - k^2\right) u = 0\,.}$$

(b) Since there is no dependence on θ, we write $u(r,\theta) = R(r)$ and the equation for R is

$$r^2 \frac{d^2 R}{r^2 r} + r\frac{dR}{dr} - \left(\frac{n\pi}{L}\right)^2 r^2 R = 0. \tag{7}$$

If the sign of the last term were positive this would be Bessel's equation with solutions $J_0(kr)$ and $Y_0(kr)$. A bit of thought shows that the solutions to Eq. (7) are Bessel functions of imaginary argument $J_0(ikr)$ and $Y_0(ikr)$. It is conventional to express the solution in terms of two function $I_0(kr)$ and $K_0(kr)$, called modified Bessel functions of the first and second kind, which are related to $J_0(ikr)$ and $Y_0(ikr)$. Hence the radial solution is

$$\boxed{R_n(r) = c_1 I_0\left(\frac{n\pi r}{L}\right) + c_2 K_0\left(\frac{n\pi r}{L}\right).}$$

(c) These have the property that for $x \to 0, I_0(x) = 1, K_0(x) = -\ln x + \text{const.}$, while for $x \to \infty, I_0(x)$ diverges exponentially, and $K_0(x)$ tends to zero exponentially. Here we need a solution which is finite at the origin, so we take $I_0(kr)$, *i.e.* $\boxed{c_2 = 0.}$

(d) The solution is then a superposition

$$T(r,z) = \sum_{n=1}^{\infty} c_n I_0\left(\frac{n\pi r}{L}\right) \sin\left(\frac{n\pi z}{L}\right).$$

Setting $r = a$ and using the boundary condition $T(a,z) = T_0 z(L-z)$ gives

$$T_0 z(l-z) = \sum_{n=1}^{\infty} c_n I_0\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi z}{L}\right),$$

which is a Fourier sine series. Evaluating the coefficients in the usual way gives

$$\begin{aligned} c_n I_0\left(\frac{n\pi a}{L}\right) &= \frac{2}{L}\int_0^L T_0 z(l-z)\sin\left(\frac{n\pi z}{L}\right) dz \\ &= \begin{cases} \dfrac{8T_0 L^2}{n^3\pi^3} & (n \text{ odd}), \\ 0 & (n \text{ even}). \end{cases} \end{aligned}$$

Hence the final solution which satisfies the boundary conditions is

$$\boxed{T(r,z) = \frac{8T_0 L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\left[\frac{(2n-1)\pi z}{L}\right]\frac{I_0[(2n-1)\pi r/L]}{I_0[(2n-1)\pi a/L]}.}$$