PHYSICS 116C
Homework 4 Solutions

1. (a)
   - Simple harmonic oscillator. Clearly the equation is of the Sturm-Liouville (SL) form with
   $$\boxed{\lambda = n^2, \quad A(x) = 1, \quad B(x) = 0, \quad w(x) = 1\,.}$$
   - Legendre's equation. Clearly the equation is of the SL form with
   $$\boxed{\lambda = n(n+1), \quad A(x) = 1 - x^2, \quad B(x) = 0, \quad w(x) = 1\,.}$$
   - Bessel's equation. This is not obviously in the SL form. To make it so, multiply by $f(x)$ and choose $f(x)$ such that the coefficient of y' is the derivative of the coefficient of y, i.e.
   $$\frac{d}{dx}\left(x^2 f(x)\right) = x f(x)$$
   or
   $$f'(x) = -\frac{f(x)}{x}$$
   which has solution $f(x) = 1/x$ and so
   $$xy'' + y' + \left[c_n^2 x - \frac{\nu^2}{x}\right] y = 0\,,$$
   which is of the SL for with
   $$\boxed{\lambda = c_n^2, \quad A(x) = x, \quad B(x) = -\nu^2/x, \quad w(x) = x\,.}$$
   - Hermite's equation. Proceeding as for Bessel's equation we multiply by $f(x)$ and require
   $$\frac{df}{dx} = -2x f(x)$$
   which has solution
   $$f(x) = \exp(-x^2)\,,$$
   and so
   $$e^{-x^2} y'' - 2x e^{-x^2} y' + 2n e^{-x^2} y = 0\,,$$
   which is of the SL form with
   $$\boxed{\lambda = 2n, \quad A(x) = e^{-x^2}, \quad B(x) = 0, \quad w(x) = e^{-x^2}\,.}$$
   - Laguerre's equation. Proceeding as for Bessel's equation we multiply by $f(x)$ and require
   $$\frac{d}{dx}(xf) = (1-x) f(x)$$
   or
   $$f' = -f\,,$$
   which has solution $f = e^{-x}$ and so
   $$x e^{-x} y'' + (1-x) e^{-x} y' + n e^{-x} y = 0\,,$$
   which is of the SL form with
   $$\boxed{\lambda = n, \quad A(x) = x e^{-x}, \quad B(x) = 0, \quad w(x) = e^{-x}\,.}$$

(b) The equations for y_1 and y_2 are

$$\frac{d}{dx}\left[A(x)y_1'\right] + \left[\lambda_1 w(x) + B(x)\right] y_1 = 0 \tag{1}$$

$$\frac{d}{dx}\left[A(x)y_2'\right] + \left[\lambda_2 w(x) + B(x)\right] y_2 = 0\,. \tag{2}$$

Multiply Eq. (1) by y_2 and Eq. (2) by y_1, subtract, and integrate from a to b. This gives

$$\int_a^b \left(y_2 \frac{d}{dx}\left[A(x)y_1'\right] - y_1 \frac{d}{dx}\left[A(x)y_2'\right] \right) dx + (\lambda_1 - \lambda_2) \int_a^b w(x) y_1(x) y_2(x)\, dx = 0\,. \tag{3}$$

The integrand in the first term can be written as

$$\begin{aligned} y_2 \frac{d}{dx}\left[A(x)y_1'\right] - y_1 \frac{d}{dx}\left[A(x)y_2'\right] &= \frac{d}{dx}\left[A(x)(y_1'y_2 - y_2'y_1)\right] - \left(A(x)y_1'y_2' - A(x)y_2'y_1'\right) \\ &= \frac{d}{dx}\left[A(x)(y_1'y_2 - y_2'y_1)\right]\,. \end{aligned}$$

Eq. (3) therefore becomes

$$\left[A(x)(y_1'y_2 - y_2'y_1)\right]_a^b + (\lambda_1 - \lambda_2) \int_a^b w(x) y_1(x) y_2(x)\, dx = 0\,.$$

Hence, for $\lambda_1 \neq \lambda_2$ we obtain the orthogonality condition

$$\boxed{\int_a^b w(x) y_1(x) y_2(x)\, dx = 0\,,} \tag{4}$$

provided the boundary term vanishes, i.e.

$$\boxed{\left[A(x)(y_1'y_2 - y_2'y_1)\right]_a^b = 0\,.} \tag{5}$$

(c)
- Simple harmonic oscillator. $\boxed{a = 0, b = 2\pi\,.}$ The y_n are $\sin nx$ and $\cos nx$ which are periodic with period 2π. $A(x) = 1$ and so $\boxed{\text{the contributions from the two limits in Eq. (5) cancel.}}$ The orthogonality condition, Eq. (4), is just the familiar orthogonality of sines and cosines over the interval from 0 to 2π.
- Legendre's equation. $\boxed{a = -1, b = 1\,.}$ The y_n are Legendre polynomials, $P_n(x)$. Eq. (5) is satisfied because $\boxed{A(x) = 1 - x^2 \text{ vanishes at } x = \pm 1\,.}$ The orthogonality condition, Eq. (4), is

$$\boxed{\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad (n \neq m)\,,}$$

  which was shown in class.
- Bessel's equation. $\boxed{a = 0, b = 1\,.}$ The y_n are Bessel functions $J_\nu(c_n x)$. In Eq. (5), the contribution at $x = 0$ vanishes because $\boxed{A(x) = x}$ and the contribution at $x = 1$ vanishes because c_n $\boxed{\text{is a zero of the Bessel function}}$ and hence $y(1) = 0$. The orthogonality condition, Eq. (4), is

$$\boxed{\int_0^1 x J_\nu(c_n x) J_\nu(c_m x)\, dx = 0 \quad (n \neq m)\,,}$$

  which was shown in class.

- Hermite's equation. $\boxed{a = -\infty, b = \infty\,.}$ The y_n are Hermite polynomials $H_n(x)$. Eq. (5) is satisfied because $\boxed{A(x) = \exp(-x^2) \text{ vanishes for } \; x \to \pm\infty\,.}$ The orthogonality condition, Eq. (4), is

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0 \quad (n \neq m)\,,}$$

which was stated in class.

- Laguerre's equation. $\boxed{a = 0, b = \infty\,.}$ The y_n are Laguerre polynomials. $L_n(x)$. Eq. (5) is satisfied because $\boxed{A(x) = xe^{-x} \text{ vanishes at } x = 0 \text{ and } \infty\,.}$ The orthogonality condition, Eq. (4), is

$$\boxed{\int_{0}^{\infty} e^{-x} L_n(x) L_m(x)\, dx = 0 \quad (n \neq m)\,,}$$

which was stated in class.

(d) Assuming completeness of the y_n we can expand a function $f(x)$ in terms of them as

$$f(x) = \sum_m a_m y_m(x)\,. \tag{6}$$

Multiplying by $y_n(x)$, integrating from a to b, and using the orthogonality relation, Eq. (4), gives

$$\int_a^b w(x) y_n(x) f(x)\, dx = a_n \int_a^b w(x) y_n^2(x)\, dx\,,$$

which gives

$$\boxed{a_n = \frac{1}{I_n} \int_a^b w(x) y_n(x) f(x)\, dx\,.} \tag{7}$$

where

$$I_n = \int_a^b w(x) y_n^2(x)\, dx\,. \tag{8}$$

(e) For the simple harmonic oscillator equation, $a = 0, b = 2\pi$, and the solutions are $\cos nx$ and $\sin nx$. Hence Eq. (7) is just the familiar Fourier series

$$f(x) = \sum_m \left(a_m \cos mx + b_m \sin mx\right)\,.$$

I_n in Eq. (8) is

$$\int_0^{2\pi} \cos^2 nx\, dx = \pi$$

or

$$\int_0^{2\pi} \sin^2 nx\, dx = \pi\,,$$

and hence Eq. (7) corresponds to

$$\boxed{a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx}$$

$$\boxed{b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx\,,}$$

which are indeed the coefficients in a Fourier series.

2. (a) Using the divergence theorem

$$\iiint_R \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \iint_B \mathbf{F} \cdot \mathbf{dS}$$

with $\mathbf{F} = u\boldsymbol{\nabla} u$ gives

$$\iiint_R \left(u\nabla^2 u + |\boldsymbol{\nabla} u|^2\right)\, dV = \iint_B u\, \boldsymbol{\nabla} u \cdot \mathbf{dS} = \boxed{\iint_B u\, \frac{\partial u}{\partial n}\, dS\,,} \tag{9}$$

where we used $\boldsymbol{\nabla} \cdot (u\mathbf{A}) = u\boldsymbol{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\nabla} u$, and $\partial u/\partial n$ is the derivative of u in the direction of the (outward) normal to the surface.

(b) Consider two solutions of Laplace's equation u_1 and u_2 which have the same values on the boundary B. Denote the difference $\boxed{u_1 - u_2 \text{ by } u}$. This satisfies Laplace's equation, $\boxed{\nabla^2 u = 0,}$ and has the value $\boxed{u = 0 \text{ on the boundary}}$. Hence inserting these properties into Eq. (9) gives

$$\iiint_R |\boldsymbol{\nabla} u|^2\, dV = 0\,. \tag{10}$$

Hence $\boldsymbol{\nabla} u = 0$, so u is a constant, in the volume V, but since $u = 0$ on the boundary S we must have $u = 0$ everywhere in V. Hence $u_1 = u_2$ and so the solution to Laplace's equation whose value is specified on the boundary $\boxed{\text{is unique.}}$

(c) Repeat the argument of the previous section but with $\partial u/\partial n = 0$ on the surface. This again yields

$$\iiint_R |\boldsymbol{\nabla} u|^2\, dV = 0\,, \tag{11}$$

showing that $\boldsymbol{\nabla} u = 0$ so u is constant. Since we do not know the value of $u \equiv u_1 - u_2$ at any point, all we can say is that the solution $\boxed{\text{is determined up to an arbitrary constant.}}$

3. (a) In one dimension, the diffusion equation is

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}\,. \tag{12}$$

Using

$$u(x,t) = \frac{1}{t^{1/2}}\, \exp\left(-\frac{x^2}{4Dt}\right) \tag{13}$$

we have

$$\boxed{\frac{\partial u}{\partial t} = \frac{-1}{2t}\, u + \frac{x^2}{4Dt^2}\, u,} \tag{14}$$

where the first term comes from differentiating $1/t^{1/2}$ and the second from differentiating the $1/t$ in the exponential. We also have

$$\frac{\partial u}{\partial x} = \frac{-x}{2Dt}\, u,$$

and differentiating again gives

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{-1}{2Dt}\, u + \left(\frac{-x}{2Dt}\right)^2 u.}$$

We see that this is just D^{-1} times Eq. (14), and hence the diffusion equation, Eq. (12), has a $\boxed{\text{solution given by by } u(x,t) \text{ in Eq. (13).}}$

(b) In spherical polar coordinates in three dimensions, the diffusion equation is

$$\frac{\partial u}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \tag{15}$$

where we assumed spherical symmetry. Taking

$$u(r,t) = \frac{1}{t^{3/2}}\,\exp\left(-\frac{r^2}{4Dt}\right) \tag{16}$$

we have

$$\boxed{\frac{\partial u}{\partial t} = \frac{-3}{2t}\, u + \frac{r^2}{4Dt^2}\, u,} \tag{17}$$

We also have

$$\frac{\partial u}{\partial r} = \frac{-r}{2Dt}\, u,$$

and taking the appropriate second derivative gives

$$\boxed{\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{-3r^2}{2Dt}\, u + r^2\left(\frac{-r}{2Dt}\right)^2 u.}$$

This is just r^2/D times Eq. (17), and hence the diffusion equation in spherical polars, Eq. (15), has a $\boxed{\text{solution given by } u(r,t) \text{ in Eq. (16).}}$

4. The wave equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}\,. \tag{18}$$

Taking

$$u(x,y,t) = \sin(k_x x)\sin(k_y y)\sin(kvt), \tag{19}$$

where $k^2 = k_x^2 + k_y^2$, we have

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -k_x^2 u,}$$

and similarly

$$\boxed{\frac{\partial^2 u}{\partial y^2} = -k_y^2 u,} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = -k^2 v^2.}$$

Hence the wave equation, Eq. (18) has a $\boxed{\text{solution given by Eq. (19).}}$

5. We have to find the solution of Laplace's equation in the rectangular region $0 \le x \le a$, $0 \le y \le b$ which satisfies the boundary conditions

$$u(0,y) = u(a,y) = u(x,0) = 0, \quad \text{and} \quad u(x,b) = u_0 \sin(\pi x/a).$$

We look for a separable solution of Laplace's equation, i.e. $u(x,y) = X(x)Y(y)$. Substituting and dividing by u gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Since the first term is only a function of x and the second term only a function of y each term must be a constant in order for the equation to be true for all x and y. Setting the first term to equal $-k^2$ we have

$$X'' + k^2 X = 0, \qquad Y'' - k^2 Y = 0.$$

A solution which satisfies the boundary conditions $u(0,y) = u(x,0) = 0$ is $X(x) = \sin(kx), Y(y) = \sinh(ky)$. To satisfy $X(a) = 0$ we need $k = 2\pi n/a$ $(n = 1, 2, 3, \cdots)$, and so we form the linear combination.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right).$$

We determine the coefficients C_n by requiring that the last boundary condition, $u(x,b) = u_0 \sin(\pi x/a)$, is satisfied.

By *inspection* we have $\boxed{C_n = 0 \ (n \neq 1),}$ and $\boxed{C_1 = u_0/\sinh(\pi b/a),}$ so

$$\boxed{u(x,y) = \frac{u_0}{\sinh\left(\frac{\pi b}{a}\right)} \sin\left(\frac{\pi x}{a}\right) \sinh\left(\frac{\pi y}{a}\right).}$$

Note: One *could* work out all the Fourier coefficients explicitly and show that they are all zero except for C_1, but this is unnecessary work. If you can see the solution by inspection, like here, then you can use this information (unless you are expressly told to go through the derivation.)

6. We look for a separable solution of Laplace's equation, i.e. $u(x,y) = X(x)Y(y)$. Substituting and dividing by u gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0.$$

Since the first term is only a function of x and the second term only a function of y each term must be a constant in order for the equation to be true for all x and y. Setting the first term to equal $-k^2$ we have

$$X'' + k^2 X = 0, \qquad Y'' - k^2 Y = 0.$$

A solution which satisfies the boundary conditions $u(0,y) = u(x,0) = 0$ is $X(x) = \sin(kx), Y(y) = \sinh(ky)$. To satisfy $X(a) = 0$ we need $k = 2\pi n/a$ $(n = 1, 2, 3, \cdots)$, and so we form the linear combination.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right).$$

To satisfy the boundary condition at $y = b$ we need

$$u_0\, x(a-x) = \sum_{n=1}^{\infty} \left[C_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right).$$

In other words we have to determine the sine Fourier series for $u_0\, x(a-x)$ in the interval $0 \le x \le a$ and the n-th Fourier coefficient is $C_n \sinh(n\pi b/a)$, *i.e.*

$$C_n \sinh\left(\frac{n\pi b}{a}\right) = u_0\, \frac{2}{a} \int_0^a x(a-x)\, \sin\left(\frac{n\pi x}{a}\right)\, dx.$$

Integrating by parts twice gives

$$\begin{aligned}
C_n \sinh\left(\frac{n\pi b}{a}\right) &= u_0 \frac{2}{a}\frac{a}{n\pi}\left\{ -\left[x(a-x)\cos\left(\frac{n\pi x}{a}\right)\right]_0^a + \int_0^a (a-2x)\cos\left(\frac{n\pi x}{a}\right)\, dx \right\} \\
&= \frac{2u_0}{n\pi}\int_0^a (a-2x)\cos\left(\frac{n\pi x}{a}\right)\, dx \\
&= \frac{2u_0}{n\pi}\frac{a}{n\pi}\left\{ \left[(a-2x)\sin\left(\frac{n\pi x}{a}\right)\right]_0^a - \int_0^a (-2)\sin\left(\frac{n\pi x}{a}\right)\, dx \right\} \\
&= \frac{4u_0 a}{(n\pi)^2}\int_0^a \sin\left(\frac{n\pi x}{a}\right)\, dx \\
&= -\frac{4u_0 a}{(n\pi)^2}\frac{a}{n\pi}\left[\cos\left(\frac{n\pi x}{a}\right)\right]_0^a = \frac{4u_0 a}{(n\pi)^2}\frac{a}{n\pi}\left[1-(-1)^n\right] \\
&= \begin{cases} 0 & (n \text{ even,}) \\ \dfrac{8a^2}{(n\pi)^3}\, u_0 & (n \text{ odd.}) \end{cases}
\end{aligned}$$

Hence the solution is (writing $n = 2m-1$ with $m = 1, 2, 3, \cdots$)

$$\boxed{u(x,y) = u_0\, a^2 \left(\frac{8}{\pi^3}\right) \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \frac{\sinh\left(\dfrac{(2m-1)\pi y}{a}\right)}{\sinh\left(\dfrac{(2m-1)\pi b}{a}\right)} \sin\left(\frac{(2m-1)\pi x}{a}\right).}$$

7. We start with

$$T(x,y) = 4T_0 \sum_{n=1}^{\infty} \frac{\sin\left[\frac{(2n-1)\pi x}{L}\right]}{(2n-1)\pi}\, e^{-(2n-1)\pi y/L}.$$

Since $\sin[(2n-1)\pi x/L] = \mathrm{Im}\,[\exp(i(2n-1)\pi x/L)]$ we can write

$$T(x,y) = \frac{4T_0}{\pi}\, \mathrm{Im} \sum_{n=1}^{\infty} \left(\frac{z^{2n-1}}{2n-1}\right), \tag{20}$$

where

$$z = \exp\left[\frac{\pi(ix-y)}{L}\right] = \exp(-y\pi/L)\left(\cos(\pi x/L) + i\sin(\pi x/L)\right).$$

Using the series for $\ln(1+z) = z - z^2/2 + z^3/3\cdots$, which also implies $\ln(1-z) = -z - z^2/2 - z^3/3\cdots$, we see that the series in Eq. (20) is just that for $\ln(1+z) - \ln(1-z) = \ln[(1+z)/(1-z)]$. Hence

$$T(x,y) = \frac{2T_0}{\pi}\, \mathrm{Im}\left[\ln\left(\frac{1+z}{1-z}\right)\right]. \tag{21}$$

For a complex number $w = r^{i\theta}$ we have $\mathrm{Im}\ln w = \mathrm{Im}\ln(re^{i\theta}) = \mathrm{Im}(\ln r + i\theta) = \theta = \arg(w) = \tan^{-1}(\mathrm{Im}w/\mathrm{Re}w)$. Letting $z = s + it$ we have

$$\frac{1+z}{1-z} = \frac{1+s+it}{1-s-it} = \frac{(1+s+it)(1-s+it)}{(1-s-it)(1-s+it)} = \frac{1-s^2-t^2+2it}{(1-s)^2+t^2},$$

and so

$$\frac{\mathrm{Im}[(1+z)/(1-z)]}{\mathrm{Re}[(1+z)/(1-z)]} = \frac{2t}{1-s^2-t^2} = \frac{2\sin(\pi x/L)e^{-\pi y/L}}{1-e^{-2\pi y/L}} = \frac{\sin(\pi x/L)}{\sinh(\pi y/L)}.$$

Hence

$$\boxed{T(x,y) = \frac{2T_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi x/L)}{\sinh(\pi y/L)}\right).} \tag{22}$$

Note:

- For $x \to 0$, $\sin(\pi x/L) \to 0$ and so $\boxed{T(0,y) = 0.}$
- For $x \to L$, $\sin(\pi x/L) \to 0$ and so $\boxed{T(L,y) = 0.}$
- For $y \to \infty$, $\sinh(\pi y/L) \to \infty$ and so $\boxed{\lim_{y\to\infty} T(x,y) = 0.}$
- For $y \to 0$, $\sinh(\pi y/L) \to 0$, so $\tan^{-1}[\sin(\pi x/L)/\sinh(\pi y/L)] \to \tan^{-1}\infty = \pi/2$ and so $\boxed{T(x,0) = T_0.}$

Hence the closed form expression in Eq. (22) satisfies the boundary conditions as required.