PHYSICS 116C
Homework 7
Due in class, Thursday November 14

1. Consider the modes of vibration of a circular drum which are circularly symmetric (*i.e.* have no dependence on θ). Denoting the frequencies of the lowest three modes by ω_1, ω_2 and ω_3 find the ratios ω_2/ω_1 and ω_3/ω_1 where ω_1 is the lowest frequency.

2. Consider a spherical shell of material of inner radius a and outer radius b. The inner surface is maintained at temperature T_1 and the outer surface at temperature T_2. Find the steady-state temperature distribution.

3. Find the steady-state temperature of a sphere of radius a when the temperature on the surface is given by
$$T(a, \theta) = \cos\theta - \cos^3\theta \,.$$
Hint: You will need to know the Legendre polynomials $P_n(x)$ for n up to 3.

4. The time-independent Schrödinger equation is
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(r)\psi = E\psi \,,$$
where $\psi(\vec{r})$ is the wavefunction, $\hbar$ is Planck's constant (divided by 2π), m is the mass, and E is the energy. Assume a spherically symmetric potential $V(r)$ which only depends on the distance r from the origin but not on direction.
   (a) Show that the angular part of the wavefunction is a spherical harmonic, $Y_l^m(\theta, \phi)$.
   (b) Show that the equation for the radial part of the wavefunction is
$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{l(l+1)\hbar^2}{2mr^2} + V(r) - E\right]R = 0 \,.$$
   (c) A particularly important example is the electron in a hydrogen atom where $V(r) = -e^2/r$ (in Gaussian units). The resulting differential equation can be solved (though you are not required to solve it here.) One finds that in order that the solution be acceptable (it must vanish at ∞ and not diverge the origin) the energy must be given by
$$E = -\frac{me^4}{2\hbar^2}\frac{1}{n^2} \,,$$
where $n = 1, 2, 3, \cdots$. Furthermore, for a given n, l can take values $0, 1, 2, \cdots, n-1$. As usual, m runs from $-l$ to l. The solution involves functions called *Associated Laguerre Polynomials.*
How many states are there for the energy level specified by quantum number n?
Note: The fact that the energy only depends on n and not on l (which denotes the *angular momentum*) is special to the $1/r$ potential. For other spherically-symmetric potentials, the energy depends on *both* l and n. This turns out to be important in atomic physics.

5. Consider a sphere of radius R of the uranium isotope ^{235}U. Let the concentration of neutrons be $n(x, t)$. Any excess concentration of neutrons tends to diffuse away, and so, in the absence of nuclear reactions, n satisfies the diffusion equation $\partial_t n = D\nabla^2 n$. However, in addition, a neutron can induce *fission* in a ^{235}U nucleus, causing it to split into two nuclei of roughly half the size plus

additional neutrons. Hence there is an additional term in the expression for $\partial_t n$ proportional to the concentration of neutrons itself. Hence we consider the following equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + cn\,,$$

where c is proportional to the probability that a neutron induces a fission per unit time.

Consider a spherically symmetric solution, and assume it separates into a product of a function of t and a function of r, *i.e.* $n(r,t) = R(r)T(t)$. Furthermore assume that n vanishes on the surface of the sphere, *i.e.* $n(R,t) = 0$.

(a) Show that $T(t)$ is given by

$$T(t) \propto e^{-\lambda t}\,.$$

(b) Show that the radial part of the solution is a spherical Bessel function

$$R(r) \propto j_0(kr)\,.$$

(c) Determine the allowed values of k and hence the allowed values of λ.

(d) Show that for small R all the allowed values of λ are positive, but that there is a critical value of the radius R_c, which you should determine, at which $\lambda = 0$.

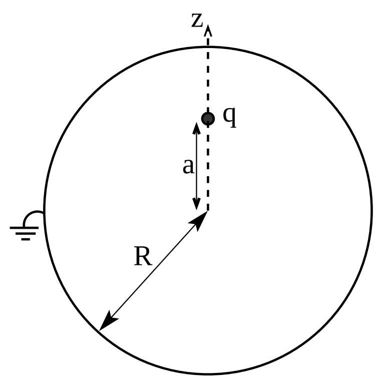
(e) What happens for $R > R_c$?

6. The electrostatic potential $V(\mathbf{r})$ satisfies Poissons equation,

$$\nabla^2 V = -\frac{\rho(\mathbf{r})}{\epsilon_0},$$

in SI units, where $\rho(\mathbf{r})$ is the charge density. Note that Poisson's equation is an *inhomogeneous* equation because the right hand side does not depend on V, but, instead, is a *given* function of $\mathbf{r}$.

Consider a point charge q at $z = a$ *inside* a grounded sphere of radius R, *i.e.* $a < R$ and $V = 0$ for $r = R$.



(a) Write down the solution for V inside the sphere as the sum of a *particular solution* including the point charge plus a solution of the corresponding *homogeneous* equation (Laplace's equation). The latter, which is called the *complementary function*, will be chosen, in the next part, such that the overall solution satisfies the boundary condition $V(r\!=\!R) = 0$. However, for now, express the complementary function as a series.

(b) Determine the coefficients in the series for the complementary function so that the solution satisfies the boundary condition at $r = R$.
Hint: It will be helpful to recall the expansion of the potential due to a point charge not at the origin in terms of Legendre polynomials. (This is, in fact, the generating function for Legendre polynomials.)

(c) Sum up the series for the complementary function to get a *closed form* expression for V.

(d) Explain how your result could have been obtained more easily by the method of images.

Note: If you have difficulty you may find it helful to go through the example in Ch. 13, Sec. 8 of Boas. (We also discussed a very similar example in class.)

7. Consider the solutions of the *homogeneous* Helmholtz equation

$$\nabla^2 u(\mathbf{r}) + \lambda^2 u(\mathbf{r}) = 0,$$

The solution will only satisfy some specified boundary conditions for discrete values of the parameter λ, which we call the "eigenvalues", λ_n. We denote the corresponding solutions by $u_n(\mathbf{r})$, and call these the "eigenfunctions". As usual, the eigenfunctions are orthogonal and normalized. Show that the corresponding Green's function, which satisfies

$$\nabla_1^2 G(\mathbf{r}_1, \mathbf{r}_2) + \lambda^2 G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{1}$$

(i.e. a particularly simple form of the *inhomogeneous* equation), with the same boundary conditions, can be written in terms of the eigenfunctions as

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{u_n(\mathbf{r}_1) u_n(\mathbf{r}_2)}{\lambda_n^2 - \lambda^2} .$$

Note: In Eq. (1), the derivatives are with respect to $\mathbf{r}_1$, while $\mathbf{r}_2$ is kept constant.
Hint: Follow the steps in the handout on Green's functions for *ordinary* differential equations. The present question shows that one there is an eigenfunction expansion for the Green's functions of PDE's as well as for ODE's.