

PHYSICS 115/242

Homework 7

Due in class, Wednesday, May 28

Note the unusual day of the week, which is because Monday is a holiday (Memorial day).

As for HW 6, put all the solutions in a single Mathematica notebook (only exception is if you do any of the analytical parts by hand).

1. Spherical Bessel functions, $j_l(x)$, and spherical Neumann functions, $y_l(x)$ frequently occur in problems of mathematical physics. They are related to ordinary Bessel and Neumann functions by

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x), \quad y_l(x) = \sqrt{\frac{\pi}{2x}} Y_{l+1/2}(x). \quad (1)$$

They also turn out to be related to trigonometric functions.

- (a) Write Mathematica functions to determine $j_l(x)$ and $y_l(x)$ in terms of trigonometric functions.
 - (b) Obtain expressions for $j_l(x)$ and $y_l(x)$ for $l = 0, 1$ and 10.
Note: Your answers should *not* have any square roots. If they do, then redefine your functions so they disappear. One possibility is to simply the expression with the assumption that $x > 0$. (The problem is that Mathematica does not simplify $\sqrt{x}\sqrt{1/x}$ to 1, because it is not true for x along the negative real axis. Strictly speaking, Eq. (1) is not valid along the negative real axis.)
Note: Starting in version 6, there are (finally!) built-in functions for spherical Bessel functions, `SphericalBesselJ[n, x]` and `SphericalBesselY[n, x]`. However, these can only be used to get numerical values; they do not express the spherical Bessel functions in terms of sines and cosines.
 - (c) Determine the first three non-vanishing terms in the series expansions of $j_{10}(x)$ and $y_{10}(x)$ about $x = 0$.
 - (d) Plot $j_1(x)$ and $y_1(x)$.
2. (a) Determine the series expansion of $\tanh(x)$ about $x = 0$ up to order 20.
(b) Use the Mathematica function, `InverseSeries` to determine, from part (a) the series for $\tanh^{-1}(x)$.
 3. In a certain approximation, called the Debye approximation, the specific heat of a solid, C , is given by

$$\frac{C}{Nk_B} = 9 \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} \frac{x^4 e^x}{(e^x - 1)^2} dx,$$

where N is the number of atoms, k_B is Boltzmann's constant, and Θ_D , called the Debye temperature, is a property of the material. Thus the Debye approximation parametrizes the specific heat in terms of a single material parameter θ_D .

- (a) Show that for $T \ll \Theta_D$,

$$\frac{C}{Nk_B} = \frac{12\pi^4}{5} \left(\frac{T}{\Theta_D} \right)^3.$$

You may use Mathematica to do the necessary integral (analytically). Insulating crystals do indeed show a T^3 specific heat at low temperatures. Note that $12\pi^4/5 \approx 234$ is a surprisingly large dimensionless number.

- (b) Show that for $T \gg \Theta_D$,

$$\frac{C}{Nk_B} = 3,$$

which is called the Dulong-Petit law. (You may either use Mathematica or pencil and paper.)

- (c) Use a one-line Mathematica command to do the integral and plot C/Nk_B against T/Θ_D .

4. In a certain approximation, called the mean field approximation, and for a certain type of “spin” (called the Ising model) the magnetization, m , of a material is given by the following equation

$$m = \tanh \left[\frac{Jm}{T} \right], \quad (2)$$

where J is a material dependent parameter (called the exchange interaction) with dimensions of temperature, and T is the temperature.

- (a) Show graphically that for $T > J$ the only solution is $m = 0$.
Note: You can use Mathematica to do this.
- (b) Show graphically that for T below J there are two other solutions, with equal and opposite magnetization.
- (c) You are *given* that when these other solutions appear they are the stable solutions. Hence the transition temperature, T_c , below which m becomes non-zero is given by $T_c = J$.
Show analytically (pencil and paper may be the most convenient) that for $T_c - T \ll T_c$

$$m^2 = \frac{3(T_c - T)}{T_c}.$$

- (d) Use a one-line Mathematica command to solve Eq. (2) and plot m versus T/T_c . You should cover a range of T/T_c from close to 0 (though not exactly equal to 0; why not?) to a little greater than 1.
- (e) On the same plot display m^2 and the approximate value for m^2 from part (c), showing that they agree just below T_c .

5. *Range of a Projectile*

In class we studied the range of a projectile in the presence of friction, assuming that the frictional force is proportional to the square of the speed. In the notebook studied in class we fixed the initial speed and varied the friction coefficient.

In this question we *fix the friction coefficient* to a value reasonable for a baseball, $k = 0.004$, and ask how far the hitter can hit the ball, x_{\max} , as a function of the initial speed v (optimized with respect to θ , the angle at which the ball is projected). Write a Mathematica function `xmax[v]`, along the lines of the handout discussed in class.

Note: We want to have `xmax` defined as a function of v so we can plot it. However, following the handout, the definition of `xmax` involves a function `xfinal[theta]` which has to be optimized with respect to `theta`. It seems that the function to be optimized in the `FindMinimum` function can *only* be a function of one variable, the one that is to be varied, so we can't, for example, make `xfinal` a function of *both* θ and v . A convenient way to pass along the value of v in the argument of `xmax` to `xfinal[theta]` is as follows:

```
xmax[vv_] := (v = vv; FindMinimum[-xfinal[theta], ... ] )
```

- (a) Determine the maximum range for $v = 50$ m/sec., and compare with the exact result in the absence of friction.
- (b) Show graphically that the effects of friction are small for small v (say $v \leq 20$ m/sec.)

Note: On the same graph plot the maximum range with and without friction. Use different types to lines to indicate which is which.

Also show graphically that the effects of friction are large at large speeds, (consider up to e.g. $v = 100$ m/sec.

Note: Have sympathy for the baseball player. To hit the ball a *little* bit further he has to hit it a *lot* harder.

6. *The Logistic Map* (This question is more challenging and so will count for more points.) Consider the logistic map

$$x_{n+1} = 4\lambda x_n(1 - x_n).$$

- (a) For each of the following values of λ : 0.2, 0.6, 0.8, 0.84, 0.90, 0.961, 0.97 and 1, determine whether successive values of x_n converge to a fixed point, a limit cycle, or show chaotic behavior.
- (b) For each of the values of λ in part (a), determine the Lyapunov exponent, λ_L . Discuss any connection you find between the values of λ_L and the nature of the trajectories in part (a).
- (c) Display graphically, as in class, region of λ which has a period-5 limit cycle.
- (d) Zoom in on one branch of the limit cycle in the last section and show that it has the familiar period doubling route to chaos.
- (e) Define λ_k as the value of λ where the period 2^{k-1} limit cycle becomes unstable and goes into a period 2^k limit cycle. We define $\lambda_0 = 1/4$ (where the fixed point appears at a non-zero value of λ) and we found in class that $\lambda_1 = 3/4$. Determine numerically (or otherwise) the values of λ_2, λ_3 and λ_4 , and hence determine $\delta_k = (\lambda_k - \lambda_{k-1})/(\lambda_{k+1} - \lambda_k)$, for several values of k . Note that the Feigenbaum constant is $\lim_{k \rightarrow \infty} \delta_k$. Compare your values for δ_k with the Feigenbaum constant.

7. Numerical Precision

Consider the sine map

$$x_{n+1} = \lambda \sin(\pi x_n)$$

for $\lambda = 0.9$.

- (a) Determine the Lyapunov exponent λ_L .
- (b) Starting with $x_0 = 0.4$ determine x_{5000} to, say five significant figures and give the precision of your answer.

Note: Don't set λ and x_0 to have decimal values (because then you are only doing calculations to machine precision). You need to specify a larger precision than that, e.g. `x0 = N[4/10, 1000]` will set the initial value of x_0 to 1000 digits precision.