PHYSICS 110A

Homework 8

Due in class, Tuesday, March 3.

1. (a) In Fig. 6.6 of Griffiths calculate the torque exerted on the square loop by the circular loop (assume r is much larger than a or b so the loops can be considered to be "pure" dipoles).

   (b) If the square loop is free to rotate what will it's final orientation be?
   Note: I suggest you take the magnetic moment of the circular loop to be in the $\hat{\mathbf{z}}$ direction and the displacement $\mathbf{r}$ from the circular loop to the square loop to be in the $\hat{\mathbf{y}}$ direction.

2. In the electrostatic case the expressions for the force $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E})$ and $\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$ are equivalent. However, this is not the case for the magnetic analogues, i.e.

$$\boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}) \neq (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B}.$$

   Explain why this is so.
   Hint: Use the expression for $\boldsymbol{\nabla}(\mathbf{A} \cdot \mathbf{B})$ given in the front cover of the book. Note that $\mathbf{m}$ and $\mathbf{p}$ do not depend on (x, y, z).
   Note: The correct expression is $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$.

3. (a) An infinitely long circular cylinder carries a uniform magnetization $\mathbf{M}$ parallel to its axis. Find the magnetic field (due to $\mathbf{M}$) inside and outside the cylinder.
   Hint: Determine the surface currents and see if you can make an analogy with a solenoid, for which we determined $\mathbf{B}$ in class.

   (b) Determine the auxiliary field $\mathbf{H}$ inside and outside the cylinder.

4. If $\mathbf{J}_f = 0$ everywhere, the curl of $\mathbf{H}$ vanishes, so we can express $\mathbf{H}$ as the gradient of a scalar potential W:

$$\mathbf{H} = -\boldsymbol{\nabla} W\,.$$

   We know that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, and using the relationship $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, we have $\boldsymbol{\nabla} \cdot \mathbf{H} = -\boldsymbol{\nabla} \cdot \mathbf{M}$. Hence

$$\nabla^2 W = (\boldsymbol{\nabla} \cdot \mathbf{M}),$$

   so W obeys Poisson's equation with $\boldsymbol{\nabla} \cdot \mathbf{M}$ as the "source".
   We can then use all the machinery described earlier for electrostatics.
   As an example, compute the field inside a uniformly magnetized sphere of radius R by solving for W using separation of variables.
   Hint: $\boldsymbol{\nabla} \cdot \mathbf{M} = 0$ everywhere except at the surface. Hence you have to solve Laplace's equation separately in the regions $r < R$ and $r > R$ (use Eq. (3.65)). The boundary conditions on the surface are (i) W is continuous (otherwise $\mathbf{H}$, which is a derivative of W be infinite on the surface) and (ii) a condition on $\partial W/\partial r$ which can be obtained from Eq. (6.24) of the book (you should be able to see how Eq. (6.24) follows from $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$).

5. Consider a coaxial cable consisting of two long cylindrical tubes, the inner one of radius a and the outer one of radius b, separated by linear insulating material of magnetic susceptibility χ_m, see Griffiths Fig. 6.24. For each tube the current distributes itself uniformly over the surface.

   (a) Find $\mathbf{H}$, and from this get $\mathbf{B}$, in the region between the tubes.

   (b) As a check, calculate the magnetization and bound currents, and check that the total current, free plus bound, gives the correct $\mathbf{B}$.

6. Consider a sphere of linear magnetic material with magnetic susceptibility χ_m in a uniform external field $\mathbf{B}_0$. Using the methods of Qu. 4 find the field inside the sphere.

7. (a) Show that the energy of a magnetic dipole in a magnetic field $\mathbf{B}$ is given by

$$U = -\mathbf{m} \cdot \mathbf{B}.$$

   (This is of the same form as the electrostatic analogue $U = -\mathbf{p} \cdot \mathbf{E}$.)
   Note: Assume that the magnitude of the dipole moment is fixed, and all you have to do is to move it into place and rotate it into its final orientation.

   (b) Show that the interaction energy of two magnetic dipoles separated by a displacement $\mathbf{r}$ is given by

$$U = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[\mathbf{m}_1 \cdot \mathbf{m}_2 - 3\,(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\,(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\right].$$

   (c) Express your answer to 7b in terms of the angles θ_1 and θ_2 that the dipoles make with $\mathbf{r}$, see Griffiths Fig. 6.30. Use your result to find the stable configuration (i.e. the one that minimizes the energy) that two dipoles would adopt if held a fixed distance apart, but left free to rotate.

   (d) Suppose you had a large collection of compass needles mounted on pins at regular intervals along a straight line. How would they point (neglecting the earth's magnetic field)?

8. At the interface between one linear magnetic material and another the magnetic field lines bend, see Griffiths Fig. 6.32. Show that

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\mu_2}{\mu_1},$$

   assuming there is no free current at the boundary.
   Note: There is a similar result for electrostatics, see Griffiths Eq. (4.68).